Cluster-Cluster Aggregation as an Analogue of a Turbulent Cascade : Kolmogorov Phenomenology, Scaling Laws and the Breakdown of Self-Similarity

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We present a detailed study of the statistical properties of a system of diffusing aggregating particles in the presence of a steady source of monomers. We emphasise the case of low spatial dimensions where strong diffusive fluctuations invalidate the mean-field description provided by standard Smoluchowski kinetic theory. The presence of a source of monomers allows the system to reach a statistically stationary state at large times. This state is characterised by a constant flux of mass directed from small to large masses. It therefore admits a phenomenological description based on the assumption of self-similarity and constant mass flux analogous to the Kolmogorov's 1941 theory of turbulence. Unlike turbulence, the aggregation problem is analytically tractable using powerful methods of statistical field theory. We explain in detail how these methods should be adapted to study the far-from-equilibrium, flux-dominated states characteristic of turbulent systems. We consider multipoint correlation functions of the mass density. By an exact evaluation of the scaling exponents for the one and two-point correlation functions, we show that the assumption of self-similiarity breaks down at large masses for spatial dimensions, $d \leq 2$. We calculate non-rigourously the exponents of the higher order correlation functions as an ϵ-expansion where $\epsilon = 2-d$. We show that the mass distribution exhibits non-trivial multiscaling. An analogy can be drawn with the case of hydrodynamic turbulence. The physical origin of this multiscaling is traced to the presence of strong correlations between particles participating in large mass aggregation events. These correlations stem from the recurrence of diffusion processes in $d \leq 2$. The analytic methods developed here will have more general applicability beyond the study of this specific problem.

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I. INTRODUCTION

The development of a coherent formalism that enables us to calculate the statistical properties of complex systems far from equilibrium remains a vexing problem in theoretical physics. It may be that the set of non-equilibrium complex systems is too diverse to admit a unified description analogous to the Gibbs formulation for equilibrium systems. Nevertheless certain subsets of non-equilibrium systems share enough common features that it is reasonable to hope that a general understanding can be obtained. One such subset which has attracted much interest, both theoretical and practical, over the years is the class of problems of turbulent type. In the present context, turbulence does not refer exclusively to hydrodynamic problems although the statistics of high Reynolds number fluid flow is a very important and challenging example. Rather, we take the word turbulent to refer to a class of non-equilibrium problems coming from a diverse range of areas in theoretical physics (from hydrodynamics to condensed matter physics to aggregation and even cosmology) sharing certain common features which we now describe.

The defining characteristic of a turbulent system is the existence of a stationary state with widely separated sources and sinks of some conserved quantity. While stationary states of turbulent systems lack detailed balance, they are characterised by a flux, mediated by nonlinear interactions, of a conserved quantity from source to sink. Universal statistics are expected in regions far from sources and sinks, also known as the inertial range. The best known example, as mentioned already, is the solution of the Navier–Stokes (N–S) equations at high Reynolds number with large scale forcing [1]. It is characterised by an energy flux from large length scales to small length scales where viscosity dissipates the energy. Other examples include Burgers turbulence [2], the Kraichnan model of passive scalar

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advection [2], kinematic magneto hydrodynamics [2], and wave turbulence [3]. A common feature of these systems is that they each admit a phenomenological description based on constancy of the flux and the assumption of self-similarity. The first such theory was the Kolmogorov 1941 (K41) theory of N–S turbulence. Obtaining a quantitative understanding of the limitations and applicability of such phenomenology is one of the major theoretical challenges. Analytic progress in hydrodynamics has been slow despite strong numerical and experimental evidence for many interesting and nontrivial violations of K41-style self-similarity. This is because the N–S equations lack an obvious small parameter that would permit a perturbative treatment. Studies of other systems, notably Burgers equation and the Kraichnan model, have been more successful [2], providing insight into the breakdown of self-similarity.

In a recent paper [4] we added to this small class of analytically tractable turbulent systems by showing breakdown of self-similarity in a system of diffusing, aggregating masses with a steady source (see section II for definition). In this paper we provide a detailed presentation of the results of [4] and the methods developed to analyse this problem. We also include several extensions of the ideas. These include the statistics of one-point moments of the mass distribution which can be shown to exhibit strong, but in a sense trivial, anomalous scaling and a section which establishes a heuristic connection between statistical field theory and an effective kinetic theory description of aggregating particles (Smoluchowski approximation). We make extensive use of statistical field theoretic techniques which, while well known in the context of equilibrium condensed matter physics, we felt would benefit from some further explanation in the context of turbulent systems. The main thrust of the calculations are explained in the main body of the article. Several appendices are provided which discuss technical details and aspects specific to the aggregation model which we study.

The layout of the article is as follows. In section II we define the "mass model" which underpins our study of aggregation. Next, in section III we begin where every study of a turbulent system should begin, namely with a simple investigation of the implications of dimensional analysis. In this section we explain the K41 phenomenology in the context of aggregation and derive the scaling properties for the mass statistics which follow from the assumption of self similarity. We then move on to the technical aspects of the article. In section IV we give a detailed explanation of how the mass model can be described in terms of an effective field theory using Doi's formalism. This can, in turn, be mapped into a variant of the $A + A \to A$ reaction diffusion model, a model which has been extensively studied. In section V we present results from numerical simulations and exact computations of the scaling exponents of the first and second order correlation functions. These confirm the breakdown of self-similarity in the mass model. The following two sections contain a detailed exploration of this breakdown using renormalisation group arguments. Section VI outlines the structure of the perturbation theory and Feynman rules for the effective field theory and outlines how to perform the computations of the relevant diagrams as expansions in $\epsilon = 2 - d$. In section VII we use reasonably standard RG arguments to derive the Callan-Symanzik equation for the n-point correlation functions. We solve these equations in $d < 2$ to calculate the anomalous scaling curve for the correlation functions as a first order epsilon-expansion. We also solve the RG equations in $d = 2$ to obtain the anomalous logarithmic corrections to the mean field answers which one expects to find at the critical dimension. In section VIII we return to a more physically-motivated discussion and outline the connection between our renormalised perturbation theory and an effective kinetic theory description based on the Smoluchowski approximation. We conclude with a brief summary of the main results and their importance. Three appendices are provided which supplement the main text with further technical explanations necessary for a complete understanding of the work. With the inclusion of the appendices we hope that the article becomes completely self contained.

II. DEFINITION OF THE MODEL

Consider a cubic lattice in d dimensions whose lattice sites are occupied by particles of integer masses. Multiple occupancy of a given site is permitted. Let the number of particles of mass m on site $\mathbf{x}$ at time t be denoted as $N_t(\mathbf{x}, m)$. At a given moment of time a configuration of the system is determined by specifying the set of occupation numbers, $\{N_t(\mathbf{x}_i, m)\}_{\mathbf{x}_i \in \mathbf{R}^d}^{m \in \mathbf{Z}^+}$. These configurations change in time due to three processes: diffusion, aggregation and input. These processes are described below.

- **Diffusion**

$$\begin{aligned} N_t(\mathbf{x}, m) &\to N_t(\mathbf{x}, m) - 1, \\ N_t(\mathbf{x} + \mathbf{n}, m) &\to N_t(\mathbf{x} + \mathbf{n}, m) + 1, \end{aligned}$$

  where $\mathbf{x} + \mathbf{n}$ denotes a nearest neighbour of site $\mathbf{x}$. This rule describes the diffusive hopping of a particle at site $\mathbf{x}$ to the neighbouring site $\mathbf{x} + \mathbf{n}$. The rate at which such hopping occurs is $DN_t(\mathbf{x}, m)/(2d)$. Here D is the diffusion constant, assumed independent of the mass, and $2d$ is the coordination number of a cubic lattice in d dimensions.

- **Aggregation**

$$\begin{aligned} N_t(\mathbf{x}, m_1) &\rightarrow N_t(\mathbf{x}, m_1) - 1, \\ N_t(\mathbf{x}, m_2) &\rightarrow N_t(\mathbf{x}, m_2) - 1, \\ N_t(\mathbf{x}, m_1 + m_2) &\rightarrow N_t(\mathbf{x}, m_1 + m_2) + 1. \end{aligned}$$

  This rule describes the aggregation of two particles of masses m_1 and m_2 at the same site $\mathbf{x}$ to form a single new particle of mass $m_1 + m_2$ at site $\mathbf{x}$. The rate at which such aggregation events occur is $\lambda N_t(\mathbf{x}, m_1) N_t(\mathbf{x}, m_2)$ where λ is the aggregation rate, also assumed independent of mass.

- **Injection**

$$N_t(\mathbf{x}, m_0) \rightarrow N_t(\mathbf{x}, m_0) + 1.$$

  This rule describes the injection of new particles of mass m_0 at random sites throughout the system. We take $m_0 = 1$. The rate of injection of new particles is J.

We will call the above model the mass model (MM). The MM has 3 physical parameters: the diffusion constant D, the aggregation rate λ, and input mass flux J. In addition there are two parameters associated with the lattice - the lattice spacing, dx, and the smallest mass, m_0.

Various aspects of this model has been studied in different contexts. The model with parallel update rules was initially studied as a simple model for river networks [5]. A comparison of different physical quantities in this model relevant for river networks can be found in [6]. The solution to the single site mass distribution in one dimension can be found in [7, 8]. From the exact solution of the two point correlation function, the exponents governing the single site mass distribution was found for arbitrary dimensions [9]. The single site mass distribution was solved for mass dependent aggregation kernel using Zakharov transformation [10] and using the Smoluchowski approximation in the context of submonolayer epitaxial thin film growth [11]. Some other contexts in which the model has been studied include granular bead packs [12], nonequilibrium phase transitions [13, 14] and self organized criticality [15]. All these studies focused on the average mass distribution and its behaviour for large masses. In this paper, we will be focussing mainly on multi-point correlation functions.

III. DIMENSIONAL ANALYSIS AND SELF-SIMILARITY CONJECTURES

A. Dimensional Considerations for the Mass Spectrum

Before doing detailed calculations, we begin by asking what we can learn about possible stationary states of the mass model from simple dimensional considerations. The first quantity of interest in characterising the long time behaviour of the model is the stationary mass spectrum, $\langle N_m \rangle$. $\langle N_m \rangle$ is the number of particles of mass m per unit volume in the stationary state. Specifically, we would like to know how $\langle N_m \rangle$ scales with m for large values of m. Since we are hoping for universal behaviour in the limit of large m, we assume that the mass spectrum does not depend on the position of the source, m_0. This assumption must be verified at a later stage. The remaining dimensional parameters upon which N_m could, in principle, depend are J, D, λ and , of course, m. We shall perform most of our computations in units where $D = 1$ but for the purposes of dimensional analysis we shall keep D.

The dimension of N_m is $[N_m] = \mathrm{M}^{-1}\mathrm{L}^{-d}$. The dimensions of the other parameters are: $[J] = \mathrm{ML}^{-d}\mathrm{T}^{-1}$, $[D] = \mathrm{L}^2\mathrm{T}^{-1}$ and $[\lambda] = \mathrm{L}^d\mathrm{T}^{-1}$. It is immediately evident that there are too many dimensional parameters in the model to uniquely determine the mass spectrum. One can readily verify that for any scaling exponent, x, and dimensionless constant, c_1, the formula

$$\langle N_m \rangle = c_1 \, J^{x-1} D^{\frac{(3-2x)d}{d-2}} \lambda^{\frac{(d+2)x-2d-2}{d-2}} m^{-x}, \tag{1}$$

is a dimensionally correct expression for $\langle N_m \rangle$ which scales as m^{-x}. This is different to the dimensional argument used by Kolmogorov in his theory of hydrodynamic turbulence. For that system, there is a unique dimensionally correct combination of parameters giving the energy spectrum. Eq. (1) allows us to pick out the scaling exponent, x, for the reaction and diffusion limited regimes:

$$x^{\mathrm{KZ}} = \frac{3}{2}, \tag{2}$$

$$x^{\mathrm{K41}} = \frac{2d+2}{d+2}. \tag{3}$$

The above two exponents correspond to a different balance of physical processes in order to realise a stationary state. We briefly discuss each to explain the choice of nomenclature.

- We shall call $x^{\rm KZ}$ the Kolmogorov–Zakharov (K–Z) exponent since it is the analogue for aggregating particles of the K–Z spectrum of wave turbulence [3] in the sense that it is obtained as the stationary solution of a mean field kinetic equation. This spectrum describes a reaction limited regime where diffusion plays no role.
- We shall call $x^{\rm K41}$ the Kolmogorov 41 (K41) exponent since it is a closer analogue of the 5/3 spectrum of hydrodynamic turbulence originally proposed by Kolmogorov in his 1941 papers on self-similarity in turbulence. This is because in the Navier-Stokes equations there is no dimensional parameter like the reaction rate controlling the strength of the nonlinear interactions. This exponent describes a diffusion limited regime where the reaction rate, λ, plays no role, reactions being effectively instantaneous.

The case $x = 1$ corresponds to one in which $\langle N_m \rangle$ does not depend on the mass flux J. However, this is not of physical interest for this problem. On the other hand, each of the regimes characterised by $x^{\rm KZ}$ and x^{K41} carry a mass flux and is relevant.

B. Self-Similarity Conjectures and Multipoint Correlation Functions

We are interested in more than just the average mass density $\langle N_m \rangle$. To characterise correlations in the the mass model we must also consider multipoint structure functions. Let $C_n(m_1, \ldots, m_n)(\Delta V)^n \prod_i dm_i$ be the probability of having particles of masses m_i in the intervals $[m_i, m_i + dm_i]$ in a volume ΔV for $i = 1 \ldots n$. $C_1(m)$ is the average mass density $\langle N_m \rangle$. We ask how $C_n(m_1, \ldots, m_n)$ varies with mass when $m_1, \ldots, m_n \gg m_0$. In particular what is the value of the homogeneity exponent γ_n defined through $C_n(\Gamma m_1, \ldots, \Gamma m_n) = \Gamma^{-\gamma_n} C_n(m_1, \ldots, m_n)$?

As for the density, dimensional analysis alone is insufficient. The formula

$$C_n(m_1, \ldots, m_n) \;=\; c_n \, J^{\gamma_n - n} D^{\frac{(3n-2\gamma_n)d}{d-2}} \lambda^{\frac{(d+2)\gamma_n + (2d+2)n}{d-2}} (m_1 \ldots m_n)^{-\frac{\gamma_n}{n}}, \tag{4}$$

is dimensionally consistent for any exponent γ_n where c_n is a dimensionless constant. We are again assuming that the large mass behaviour of the C_n's is independent of m_0. The simplest way to obtain a theoretical prediction for the mass dependence of the C_n's is to use a self-similarity conjecture similar to Kolmogorov's 1941 conjecture about the statistics of velocity increments in hydrodynamic turbulence. Assume that C_n depends only on the masses m_i, mass flux J and either the reaction rate, λ or the diffusion coefficient D. Depending on which assumption we make, dimensional analysis allows us to determine the mass dimension of C_n. For the reaction limited case we obtain

$$\gamma_n^{\rm KZ} = \frac{3}{2} n, \tag{5}$$

and for the diffusion limited case,

$$\gamma_n^{\rm K41} = \frac{2d+2}{d+2} n. \tag{6}$$

Note that in both cases, the dependence of γ_n on the index n is linear, reflecting the assumed self-similarity of the statistics of the local mass distribution. When $n = 1$, $\gamma_1^{\rm K41} = (2d+2)/(d+2)$ coincides with the result of an exact computation of γ_1 for $d < 2$ [9] so we expect that the K41 conjecture is the appropriate theory in $d < 2$. In $d > 2$ it is known that $\gamma_1 = 3/2$, hence $\gamma_1^{\rm KZ} = 3/2$ is the correct scaling for the density. Therefore the KZ conjecture is appropriate in higher dimensions. For $d > 2$, the statistics of the MM should be accurately described by mean field theory (KZ) and the self-similarity conjecture should hold. In this paper we will concern ourselves with $d \leq 2$.

The K41 self-similarity conjecture assumes that C_n does not depend on λ, m_0, the lattice spacing, and the box size $\Delta V dm_1 \ldots dm_n$. The lack of dependence on the lattice spacing is expected due to the renormalizability of the effective field theory describing the MM below two dimensions. We will however find an anomalous dependence of correlation functions on a length scale depending on the other parameters and the box size which leads to a violation of self-similarity.

IV. FIELD THEORETIC DESCRIPTION OF THE MASS MODEL

A. Motivation

In order to check the validity of Kolmogorov conjecture for the mass model for $n > 1$, we need to go beyond dimensional analysis. We shall do this by constructing an effective field theory that provides us with a continuum description of the model. We are helped by the fact that the effective field theory which describes the mass model is closely related to a much simpler and well studied theory which describes the $A+A \to A$ reaction-diffusion model. We shall then use standard techniques of statistical field theory to extract information about the mass model from this theory. As is well known, this model has critical dimension 2. In dimensions greater than 2, a mean field description, characterised by K–Z scaling works. In $d \leq 2$, this mean field description breaks down and correlations between particles become important. We shall show how to take into account these correlations and we demonstrate how they lead to the breakdown of self-similarity in $d \leq 2$ by calculating the scaling behaviour of the correlation functions, C_n, for large masses. Unlike for the case hydrodynamic turbulence, such an analysis is possible for the mass model. This is because in $d \leq 2$, the large mass statistics of the model in are governed by a perturbative fixed point of the renormalization group flow in the space of coupling constants of the model. The order of this fixed point is $\epsilon = 2 - d$, which allows one to compute the relevant scaling exponents in the form of an ϵ-expansion.

B. Effective Action for the Mass Model

Using Doi's formalism, it is possible to construct an effective field theory of the mass model. The steps in the procedure are as follows:

1. Write a *master equation* for time evolution of $\mathbf{P}(\{N_t(\mathbf{x}_i, m)\})$, the probability of finding the system in a given configuration, $\{N_t(\mathbf{x}_i, m)\}$. The master equation is linear and first order in time.

2. Introduce creation and annihilation operators, $a_{i,m}$ and $a^\dagger_{i,m}$, which create and destroy particles of mass m at site $\mathbf{x}_i$. Then convert the master equation into a Schroedinger equation :

$$\frac{d}{dt}\,|\psi(t)\rangle = -H[a_{i,m}, a^\dagger_{i,m}]\,|\psi(t)\rangle\,,$$

   using Doi's formalism (second quantisation) [16, 17].

3. Use the Feynman trick to derive a functional integral measure which converts the second quantised Schroedinger equation into a continuous field theory.

These steps are well described in the context of reaction-diffusion models in [18, 19]. For the model of interest here, the procedure is similar to the reaction-diffusion case with the algebra made slightly more complicated by the necessity to keep track of a mass index for each particle. Explicit formulae for the master equation and Hamiltonian operator of the MM are given in appendix A.

After going over to a path integral formulation of the Schroedinger equation we can express the average density and other correlation functions as path integrals. Further detail on this procedure can be found in appendix B. For example the average density, $N_t(m) = \langle \phi_{\mathbf{x},m}(\tau) \rangle$, is given, in the notation of appendix B, by

$$N_t(m) = \int \mathcal{D}\phi \mathcal{D}\phi^* \phi_{\mathbf{x},m}(\tau)\, e^{-S_{\rm MM}[\phi,\phi^*,D,J,t,\lambda]}, \tag{7}$$

where

$$\begin{aligned} S_{\rm MM}[\phi,\phi^*,D,J,t,\lambda] \;=\; & \int_0^t d\tau \int d^d\mathbf{x}\, dm\;\{\phi^*\partial_t\phi \\ & + H[\phi,\phi^*]\}\,, \end{aligned} \tag{8}$$

and

$$\begin{aligned} H[\phi,\phi^*] \;=\; & D\,\nabla_{\mathbf{x}}\phi^*_m \cdot \nabla_{\mathbf{x}}\phi_m + \frac{J}{m}\delta(m-m_0)\phi^*_m \\ & - \lambda \int dm_1 dm_2\,\{\delta(m_2 - m - m_1) \\ & \left[\phi^*_{m_2} - 2\phi^*_m - \phi^*_m\phi^*_{m_1}\right]\phi_m\phi_{m_1}\Big\}\,. \end{aligned}$$

C. Dimensional Analysis of the Effective Action

As for the case of stochastic aggregation without source [20], it is helpful to nondimensionalise the fields ϕ and ϕ^* and express the action in eq. (8) solely in terms of dimensionless quantities. Introduce dimensionless fields, $\bar{\phi}$ and $\bar{\phi}^*$, in eq. (7) by the following rescalings:

$$\begin{aligned} \tau &\rightarrow t\,\tau, \\ \mathbf{x} &\rightarrow \sqrt{Dt}\,\mathbf{x}, \\ m &\rightarrow \lambda J t^2\, m, \\ \phi &\rightarrow \frac{1}{\lambda^2 J t^3}\,\bar{\phi}, \\ \phi^* &\rightarrow \bar{\phi}^*, \end{aligned}$$

to obtain

$$N_t(m) = \int \mathcal{D}\phi\mathcal{D}\phi^*\phi_{\mathbf{x},m}(\tau)\, e^{-\frac{1}{g}S_{\mathrm{MM}}[\bar{\phi},\bar{\phi}^*,1,1,1,1]}, \tag{9}$$

where the dimensionless interaction coefficient is

$$g = D^{\frac{d}{2}}\, t^{\frac{d-2}{2}}\, \lambda. \tag{10}$$

The fact that $g \rightarrow 0$ as $t \rightarrow \infty$ for dimensions greater than 2 and $g \rightarrow \infty$ as $t \rightarrow \infty$ for dimensions less than 2 expresses the well known fact that the critical dimension of the mass model is 2. We shall have much more to say about this later.

D. The Stochastic Smoluchowski Equation

It is possible to establish an exact map between the field theory in eq. (7) and the following stochastic integro-differential equation, [19, 20]:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - D\nabla^2\right)\phi(m) &= \lambda\int_0^m dm'\phi(m')\phi(m-m') \\ -2\lambda\phi(m)N + \frac{J}{m_0}\delta(m-m_0) &+ i\sqrt{2\lambda}\phi(m)\eta(\vec{x},t), \end{aligned} \tag{11}$$

where $N = \int_0^\infty dm'\phi(m')$, $i = \sqrt{-1}$, and $\eta(\vec{x},t)$ is white noise in space and time:

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle = \delta(t-t')\delta^d(\vec{x}-\vec{x}'). \tag{12}$$

The technical details of this mapping can be found in appendix B. Without the noise term, one recognises eq. (11) as the mean field (Smoluchowski) equation of the model. Thus, all fluctuation effects are encoded in the imaginary multiplicative noise term.

E. Correspondence with $A + A \rightarrow A$ Model

Eq. (11) simplifies after taking Laplace transform with respect to the mass variable [20]. Let

$$R_\mu(\vec{x},t) = \int_0^\infty dm\phi(\vec{x},m,t) - \int_0^\infty dm\phi(\vec{x},m,t)e^{-\mu m}. \tag{13}$$

Then,

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)R_\mu(\vec{x},t) = -\lambda R_\mu^2 + \frac{j_\mu}{m_0} + 2i\sqrt{\lambda}R_\mu(\vec{x},t)\eta(\vec{x},t), \tag{14}$$

where $j_\mu = J(1-e^{-\mu m_0})$. In terms of field $R_\mu(\vec{x},t)$, eq. (14) becomes a stochastic version of the rate equation for the $A + A \rightarrow A$ reaction in the presence of a source. Hence, the computation of the average mass distribution in the mass

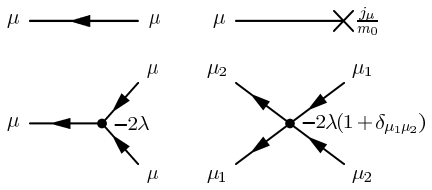

FIG. 1: Propagators and vertices of the theory.

model reduces to solving a one-species particle system with a μ-dependent source and then computing the inverse Laplace transform with respect to μ. For example, to compute the average density, $\langle N_m(t)\rangle$, for the mass model, we first calculate $\langle R_\mu\rangle$, the average of the solution of eq. (14) with respect to the noise, $\eta(\vec{x},t)$. We then take the inverse Laplace transform with respect to μ and obtain the density from eq. (13). By applying the Martin–Siggia–Rose (MSR) procedure [21] to eq. (14), we can write $\langle R_\mu\rangle$ as a functional integral :

$$\langle R_\mu\rangle = \int \mathcal{D}R_\mu \mathcal{D}\widetilde{R}_\mu \; R_\mu \, e^{-S_{\rm RD}[R_\mu,\widetilde{R}_\mu]}, \tag{15}$$

where the effective action for the reaction-diffusion system described by eq. (14) is

$$\begin{aligned} S_{\rm RD}[R_\mu,\widetilde{R}_\mu] \;&=\; \int d\mathbf{x}dt \left[\widetilde{R}_\mu(\partial_t - D\Delta)R_\mu + \lambda\widetilde{R}_\mu R_\mu^2 \right. \\ &\left. +\lambda\widetilde{R}_\mu^2 R_\mu^2 - \frac{j}{m_0}\widetilde{R}_\mu\right]. \end{aligned} \tag{16}$$

In order to compute higher order correlation functions $C_n(m,t)$, we need to know correlation functions of the form $\langle R_{\mu_1}(\vec{x},t)R_{\mu_2}(\vec{x},t)\ldots R_{\mu_n}(\vec{x},t)\rangle$. These are non-trivial, as the stochastic fields $R_\mu(\vec{x},t)$'s are correlated for different values of μ via the common noise term in eq. 14. To clarify what is meant by this, we apply the MSR procedure to two copies of eq. (14) describing the evolution of R_{μ_1} and R_{μ_2} respectively to obtain a functional integral representation for $\langle R_{\mu_1}R_{\mu_2}\rangle$. This gives :

$$\begin{aligned} \langle R_{\mu_1}R_{\mu_2}\rangle \;&=\; \int \mathcal{D}R_{\mu_1}\mathcal{D}\widetilde{R}_{\mu_1}\mathcal{D}R_{\mu_2}\mathcal{D}\widetilde{R}_{\mu_2} \; R_{\mu_1}R_{\mu_2} \\ &\times e^{-S_{\rm RD}[R_{\mu_1},\widetilde{R}_{\mu_1}]}\, e^{-S_{\rm RD}[R_{\mu_2},\widetilde{R}_{\mu_2}]} \\ &\times e^{-\int d\mathbf{x}dt \; 2\lambda\widetilde{R}_{\mu_1}\widetilde{R}_{\mu_2}R_{\mu_1}R_{\mu_2}}. \end{aligned} \tag{17}$$

The point to note here is that the path integral measure for correlations of R_μ for *different* values of μ does not factorise owing to the presence of the last term in eq. (17). Thus, to compute n-point correlation functions in the MM, one needs to analyse a system of n stochastic rate equations of $A+A\to A$ theory coupled via common noise terms of the form shown in eq. (17). Some detailed explanation of the physical interpretation of higher order correlation functions in terms of the probability of multi-particle configurations in the particle system is provided in appendix C.

F. Feynman Rules

Solving eq. (14) perturbatively in λ and j, and then averaging over noise, one can derive the set of Feynman rules for the computation of correlation functions. Alternatively they can be written down directly from the action, eq. (16). See [18] for the details of the procedure. However care must be taken to include the "extra" vertex which arises when computing correlations between fields with different μ indices. The Feynman rules are summarised in fig. 1 with time increasing from right to left. The slightly more complicated prefactor for the quartic vertex takes into account the aforementioned "extra" vertex.

The propagator is just the regular diffusive Green's function which, in d spatial dimensions, is

$$G_0(\mathbf{x}_2-\mathbf{x}_1,t_2-t_1) = 4\pi(t_2-t_1)^{-\frac{d}{2}} \; e^{-\frac{|(\mathbf{x}_2-\mathbf{x}_1|}{4(t_2-t_1)}}, \tag{18}$$

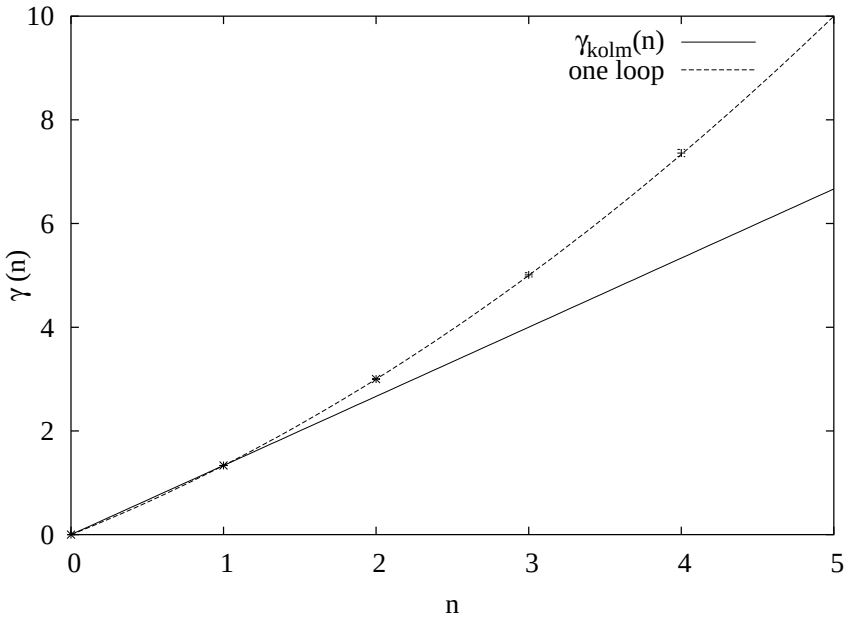

FIG. 2: γ_n as a function of n is one dimension. The straight line shows the Kolmogorov answer [eq. (6)]. The dotted line shows eq. (74) with $\epsilon = 1$ and terms of order ϵ^2 and higher set to zero. The values γ_0, γ_1 and γ_2 are exact. γ_3 and γ_4 were obtained by Monte Carlo simulations performed on a lattice of size 10^5 and averaged over 2×10^7 Monte Carlo time steps with $J = 4D$.

or

$$\hat{G}_0(\mathbf{k}, t) = (2\pi)^{-\frac{d}{2}}\ e^{-k^2 t}, \tag{19}$$

in the momentum-time representation usually used in computations.

The one-point function, $\langle R_\mu \rangle$ is then given by the sum of all diagrams constructed from the building blocks shown in fig. 1 with a single outgoing line. Likewise, the n-point correlation function $\langle R_{\mu_1}(\vec{x}_1, t_1)\ R_{\mu_2}(\vec{x}_2, t_2) \ldots R_{\mu_n}(\vec{x}_n, t_n)\rangle$ is given by the sum of the contributions of all diagrams which have n-outgoing lines. In section VI we shall turn to actual computations.

V. NUMERICAL SIMULATIONS AND MULTISCALING

A. Numerical Simulations of the Mass Model in d=1

We first look at the results of Monte Carlo simulations of the MM which confirm that there is indeed some interesting behaviour which requires explanation. In particular, numerical simulations show a breakdown of self-similarity in the mass model in one dimension and multiscaling of the correlation functions, C_n. The results are shown in fig. 2.

B. Constant Flux Relation - Analytic Confirmation of Multiscaling

We know that the K41 hypothesis works for $n = 1$. From fig. 2, it is clear that the Kolmogorov scaling breaks down for $n > 1$. It is also possible to analytically confirm that $\gamma_n \neq \gamma_n^{K41}$ in $d < 2$ by computing γ_2. From the definition of γ_2, if follows that

$$\Phi_2(m_1, m_2) = \left(\frac{1}{m_1 m_2}\right)^{\gamma_2/2} \phi\left(\frac{m_1}{m_2}\right), \tag{20}$$

where ϕ is an unknown scaling function which satisfies $\phi(x) = \phi(1/x)$ due to a symmetry. Our aim is to compute γ_2 without using the ϵ-expansion which we shall use in section VII to compute γ_n for general n.

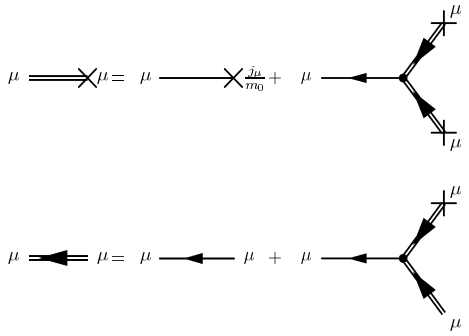

FIG. 3: Diagrammatic form of mean field equations for $R_\mu^{\rm mf}(t)$ and the tree-level propogator, $G_\mu^{\rm mf}(\mathbf{2};\mathbf{1})$.

As we are interested in $\Phi_2(m_1,m_2)$ for $m_1,m_2>0$,

$$\Phi_2(m_1,m_2)=\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} d\mu_1 d\mu_2 \langle R_{\mu_1}R_{\mu_2}\rangle, e^{-m_1\mu_1}\, e^{-m_2\mu_2}, \tag{21}$$

where R_μ solves eq. (14). Due to eq. (20),

$$\langle R_{\mu_1}R_{\mu_2}\rangle=\left(\frac{1}{\mu_1\mu_2}\right)^{1-\gamma_2/2}\psi\left(\frac{\mu_1}{\mu_2}\right), \tag{22}$$

where ψ is an unknown scaling function. To find the large m_1,m_2 asymptotics of Φ, we need to know the small μ_1,μ_2 asymptotics of $\langle R_{\mu_1}R_{\mu_2}\rangle$. Averaging eq. (14), with respect to noise and setting $\partial_t\langle R_\mu\rangle=0$ in the large time limit, we find that $\langle R_\mu R_\mu\rangle=\frac{j}{\lambda m_0}\approx\frac{J\mu}{\lambda}$ for $\mu\ll m_0$. Comparing this result with eq. (22) we find that $\gamma_2=3$.

Note that γ_2 does not depend on dimension, d, of the lattice. Therefore, it is correctly predicted by mean field theory. The non-renormalization of γ_2 by diffusive fluctuations can be explained by mass conservation or, more precisely by constancy of the average flux of mass in the mass space, see [10] for more details. Here, we simply wish to point out that the exact answers for γ_1 and γ_2 establish multiscaling non-perturbatively: the points $(0,0)$, $(1,\gamma_1)$ and $(2,\gamma_2)$ do not lie on the same straight line.

Due to its close connection with mass conservation, the law $\gamma_2=3$ is a counterpart of the 4/5 law of Navier-Stokes turbulence. Recall, that 4/5 law states that the third order longitudinal structure function of the velocity field scales in the inertial range as the first power of the separation. It is interesting to notice, that Kolmogorov theory respects 4/5 law in Navier-Stokes turbulence, but violates $\gamma_2=3$ in the MM.

VI. PERTURBATIVE EXPANSION FOR CORRELATION FUNCTIONS

A. Mean Field Analysis

The mean field theory associated with the field theory described by the effective action, eq. (8) can be thought of in several complementary ways. Let us suppose that the reaction rate, λ, is the smallest parameter in the problem. This means that the dimensionless interaction coefficient, g, given by eq. (10) is small. In this case, the path integral in eq. (9) can computed in the limit $g\to 0$ using the saddle point method. In this limit, ϕ satisfies the Euler-Lagrange equation (expressed in dimensional variables) :

$$\left(\frac{\partial}{\partial t}-D\nabla^2\right)\phi(m)=\lambda\int_0^m dm'\phi(m')\phi(m-m')$$
$$-2\lambda\phi(m)N+\frac{J}{m_0}\delta(m-m_0), \tag{23}$$

which we recognise as the mean field equation derived for classical aggregation problems by Smoluchowski. Now, if $g\to 0$ then it follows that the noise term disappears from the non-dimensionalised version of the stochastic Smoluchowski equation, eq. (11), leaving us with a deterministic equation for the density, which is again the classical

Smoluchowski equation. For readers interested in the analogy between stochastic aggregation and wave turbulence, the mean field Smoluchowski equation is the analogue of the kinetic equation.

In [10] we studied in great detail the stationary state of eq. (23) and showed that the spectrum

$$N_m = \sqrt{\frac{J}{4\pi\lambda}}\, m^{-\frac{3}{2}}, \tag{24}$$

is the exact stationary solution as $t \to \infty$. This solution carries a constant flux, J, of mass from small masses to large. This is the Kolmogorov– Zakharov spectrum of the mass model which we identified from dimensional considerations in section III as corresponding to a reaction limited regime.

When do we expect the mean field answers to be correct? The K–Z solution is established in the limit of large times. Since the mean field results become exact in the limit $g \to 0$, eq. (10) implies that the spectrum (24) should be correct for $d > 2$. For $d \leq 2$ the mean field approximation quickly breaks down and we must take into account the effect of fluctuations. This will be the main objective of the rest of this article.

Let us now identify clearly the terms in the diagrammatic expansion which give the mean field answers so that we can see how to use our formalism to compute the fluctuations about the mean field. Since the mean field kinetic equation corresponds to the deterministic limit of the stochastic Smoluchowski equation, the corresponding field theory has no loops. Therefore we expect the mean field answers for the average density to correspond to the sum of all tree diagrams with a single outgoing line. Let us now analyse these.

Let R_{mf}, denoted by a thick line with a cross, be the contribution to R from all tree level diagrams. The equation satisfied by R_{mf} is shown is diagrammatic form in fig. 3A. In equation form, it reads

$$\frac{dR_{mf}}{dt} = \frac{j_\mu}{m_0} - \lambda R_{mf}^2. \tag{25}$$

This is easily solved to give

$$R_{mf}(t) = \sqrt{\frac{j_\mu}{m_0\lambda}} \tanh\left(\sqrt{\frac{j_\mu\lambda}{m_0}}t\right) \stackrel{t\to\infty}{\longrightarrow} \sqrt{\frac{j_\mu}{m_0\lambda}}. \tag{26}$$

Performing the inverse Laplace transform in the limit $\mu \to 0$ we find that as $m \to \infty$,

$$N_m \sim \sqrt{\frac{J}{4\pi\lambda}}\, m^{-\frac{3}{2}},$$

and recover the K–Z spectrum as we should. Both the constant and the exponent agree with those obtained by the Zakharov transformation of the mean field kinetic equation confirming that our approach makes sense.

It is convenient to define $G_\mu^{\rm mf}(x_2t_2; x_1t_1)$ as the propagator that includes all the tree level diagrams. The equation obeyed by it is shown in fig. 3B. The solution is

$$G_\mu^{\rm mf}(\mathbf{2};\mathbf{1}) = G_0(\mathbf{2};\mathbf{1}) \left[\frac{\cosh\sqrt{\frac{j\lambda}{m_0}}t_1}{\cosh\sqrt{\frac{j\lambda}{m_0}}t_2}\right]^2, \tag{27}$$

$$\stackrel{t_{1,2}\to\infty}{\longrightarrow} G_0(\mathbf{2};\mathbf{1}) e^{-\Omega(t_2-t_1)}, \tag{28}$$

where G_0 is the Green's function of the linear diffusion equation, eq. (19), and

$$\Omega_\mu = 2\sqrt{\frac{j_\mu\lambda}{m_0}}, \tag{29}$$

is the inverse of the mean field response time.

B. Loop Expansion

In order to take into account fluctuations about the mean field answer we need to compute diagrams with loops. Ordering the terms in the perturbation series according to the number of loops is known as a *loop expansion*. Using the mean field density and response functions computed above simplifies the task of computing the sum of all diagrams

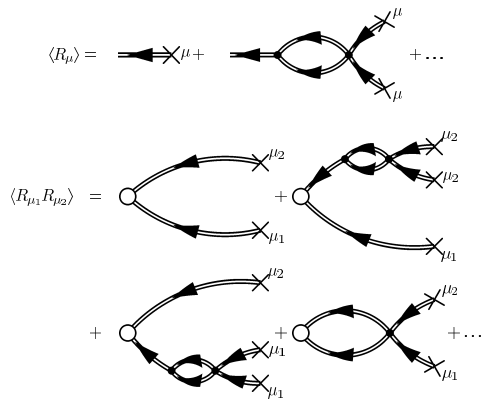

FIG. 4: Zeroth and first order terms in the loop expansions for $\langle R_\mu \rangle$ and $\langle R_{\mu_1} R_{\mu_2} \rangle$

with a given number of loops. We now demonstrate by power counting that loop expansion of the mean mass distribution corresponds to weak coupling expansion with respect to λ. The quantity $\langle R(\vec{x},t) \rangle$ is given by the sum of all diagrams with one outgoing line built out of blocks shown in fig. 1. Consider such a diagram containing L loops, V vertices and N R_{mf}-lines. The corresponding Feynman integral contains (in the mixed momentum-time representation) dL momentum integrals and V time integrals. Hence, the integration over all times and momenta produces the factor $\Omega^{-V+dL/2} \sim \lambda^{-\frac{V}{2}+\frac{dL}{4}}$. The N R_{mf}-lines produce the factor $R_{mf}^N \sim \lambda^{-N\lambda/2}$. A factor λ^V comes from V vertices of the graph. Hence the corresponding Feynman integral is proportional to $\lambda^{-\frac{N}{2}+\frac{V}{2}+\frac{dL}{4}}$. Note also that the number of triangular vertices in the graph is equal to $N-1$ and the number of quadratic vertices is equal to the number of loops L. Thus the total number of vertices is given by $V = L + N - 1$. Therefore, any L-loop graph contributing to the average mass distribution is proportional to $\lambda^{-\frac{1}{2}+\frac{L}{2}(1+\frac{d}{2})}$. We conclude that loop expansion corresponds to the perturbative expansion of R around the mean field value with the parameter $\lambda^{\frac{2+d}{4}}$.

C. Breakdown of Loop Expansion

The conditions under which the loop corrections to the mean field answer can be neglected are most simply derived using dimensional analysis. The scale of diffusive fluctuations is given by the only constant of dimension length which can be constructed out of μ and J: $L_D = (\mu J)^{-1/(d+2)}$. The dimensionless expansion parameter in the loop expansion above is $g_0(\mu) = \lambda L_D^{\epsilon/2}$, where $\epsilon = 2 - d$. The large mass behaviour of N_m is determined by the small-μ behaviour of R_μ. In $d < 2$, g_0 goes to infinity in the limit $\mu \to 0$ and the loop expansion breaks down. Thus a re-summation of the loop expansion is needed in order to extract the large-m behaviour of N_m in low dimensions.

D. Calculation of One Loop Corrections to Mean Field Theory

Let us now compute $\langle R_\mu \rangle$ and $\langle R_{\mu_1} R_{\mu_2} \rangle$ to one loop order. The diagrams are shown in fig. 4. The corresponding algebraic expressions are evaluated by dimensional regularisation in dimension d, no longer necessarily an integer.

$$\begin{aligned} \langle R_\mu \rangle &= R_\mu^{\mathrm{mf}} + R_\mu^{(1)} + \dots \\ &= \sqrt{\frac{j_\mu}{\lambda m_0}} \left[1 + \frac{\lambda^2 \Omega_\mu^{\frac{d-4}{2}}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \sqrt{\frac{j_\mu}{\lambda m_0}} + \dots \right], \end{aligned} \tag{30}$$

where ... represent terms of higher order in λ which necessarily have more loops. In this formula we have introduced the quantity ϵ, defined as

$$\epsilon = 2 - d, \tag{31}$$

to measure the deviation of the dimension of the system from the critical dimension.

The four diagrams for $\langle R_{\mu_1} R_{\mu_2} \rangle$, shown in fig. 4, give the following respective contributions :

$$\begin{aligned} \langle R_{\mu_1} R_{\mu_2} \rangle &= R^{\rm mf}_{\mu_1} R^{\rm mf}_{\mu_2} - \frac{2\lambda}{(8\pi)^{\frac{d}{2}}} \frac{R^{\rm mf}_{\mu_1} R^{\rm mf}_{\mu_2}}{(\Omega_{\mu_1}+\Omega_{\mu_2})^{\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \\ &\quad + R^{(1)}_{\mu_1} R^{\rm mf}_{\mu_2} + R^{\rm mf}_{\mu_1} R^{(1)}_{\mu_2} \ldots, \\ &= \langle R_{\mu_1} \rangle \langle R_{\mu_2} \rangle \left[1 - \frac{2\lambda}{(8\pi)^{\frac{d}{2}}} \frac{1}{(\Omega_{\mu_1}+\Omega_{\mu_2})^{\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) + \ldots \right], \end{aligned} \tag{32}$$

an expression which is correct to one loop order. Note that the second diagram in the expression for $\langle R_{\mu_1} R_{\mu_2} \rangle$ describes the correlation between R_μ fields for different values of μ and prevents the factorisation of the 2-point function into a product of 1-point functions. We shall need these expressions again to when we use RG to resum the loop expansion.

For a given mass scale, m, there is a corresponding μ scale, $1/m$ and a corresponding length scale, L_μ defined as

$$L_\mu = \left(\frac{j_\mu}{D m_0} \right)^{-\frac{1}{d+2}}. \tag{33}$$

At this point, let us also define the dimensionless reaction rate, g, as

$$g = \lambda L_\mu^\epsilon. \tag{34}$$

In what follows, it shall be convenient to express eq. (30) and eq. (32) in terms of L_μ and g. This gives

$$\langle R_\mu \rangle = L_\mu^{\epsilon-2} \frac{1}{\sqrt{g}} \left[1 + \frac{1}{4(2\pi)^{1-\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) g^{1-\frac{\epsilon}{4}} + \ldots \right], \tag{35}$$

and

$$\begin{aligned} \langle R_{\mu_1} R_{\mu_2} \rangle &= \langle R_{\mu_1} \rangle \langle R_{\mu_2} \rangle \left[1 - \frac{g^{1-\frac{\epsilon}{4}}}{(4\pi)^{1-\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \right. \\ &\quad \left. \times \left(\left(\frac{L_{\mu_1}}{L_\mu} \right)^{\frac{\epsilon-4}{2}} + \left(\frac{L_{\mu_2}}{L_\mu} \right)^{\frac{\epsilon-4}{2}} \right)^{-\frac{\epsilon}{2}} + \ldots \right]. \end{aligned} \tag{36}$$

Note that the μ dependence of this expression is illusory since g also depends on μ. To study the behaviour of the corrections to mean field answers which we have just calculated, we need to study the large m behaviour of the Laplace transforms with respect to the μ's of the expressions in eq. (30) and eq. (32). Simple calculation shows that the second terms inside the square brackets in these expressions diverge as the μ's are taken to zero when $\epsilon > 0$ signifying a breakdown of the loop expansion. This is as expected from the power counting argument of section VI C.

VII. RENORMALISATION GROUP ANALYSIS FOR $d < 2$

A. Epsilon Expansion

The loop expansions for correlation functions computed in section VI D are expansions in powers of the dimensionless reaction rate, g. The problem is that in $d < 2$, these expansions become badly ordered as we approach $L_\mu \to \infty$. However, since we have computed correlation functions in arbitrary dimension, d, we can convert the loop expansions into expansions in $\epsilon = 2 - d$ at a fixed value of g. For $\epsilon \ll 1$, eq. (35) and eq. (36) can be written :

$$\langle R_\mu \rangle = L_\mu^{\epsilon-2} \frac{1}{\sqrt{g}} \left[1 + \frac{g}{4\pi\epsilon} + \ldots \right], \tag{37}$$

and

$$\langle R_{\mu_1} R_{\mu_2} \rangle = \langle R_{\mu_1} \rangle \langle R_{\mu_2} \rangle \left[1 - \frac{g}{2\pi\epsilon} + \ldots \right]. \tag{38}$$

FIG. 5: Diagrams contributing to the renormalisation of the reaction rate.

Of course these series are still badly ordered as $\epsilon \to 0$. The idea is to replace certain correlation functions with appropriate renormalised quantities, also expressed as expansions in ϵ, such that the renormalised counterparts of the above expressions are well-ordered in ϵ. The final pay-off comes when we find that these expressions remain well ordered in ϵ even when we take the limit $L_\mu \to \infty$ because of the presence of a *perturbative fixed point*, a structural feature of the theory which we must now explain in order to make sense of this scheme.

B. Renormalised Reaction Rate and β-function

The presence of a perturbative fixed point for the $A + A \to A$ model was originally pointed out by Peliti [22]. The corresponding calculations in the presence of a source were done by Droz[23]. This is sufficient to deal with the problem at hand. Nevertheless we shall paraphrase their arguments here for the sake of completeness.

Let us define a renormalised reaction rate, λ_R, as the amputated 3-point vertex function shown diagrammatically in fig. 5. After performing the algebra we find

$$\lambda_R = \lambda \left[1 - \frac{1}{2(2\pi)^{\frac{d}{2}}} \Gamma\left(\frac{\epsilon}{4}\right) g^{1-\frac{\epsilon}{4}} + \ldots \right]. \tag{39}$$

Now we introduce a dimensionless renormalised reaction rate, $g_R = \lambda_R L_\mu^\epsilon$, as we did in eq. (34), which is given by

$$g_R = g - \frac{1}{2(2\pi)^{1-\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{4}\right) g^{2-\frac{\epsilon}{4}} + \ldots. \tag{40}$$

For small values of ϵ this can be written as

$$g_R = g - g_*^{-1}(\epsilon) g^2 + \ldots, \tag{41}$$

where

$$\begin{aligned} g_*^{-1}(\epsilon) &= \frac{1}{2(2\pi)^{1-\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{4}\right) \\ &= \frac{1}{2\pi\epsilon} + o(1) \qquad \epsilon \ll 1. \end{aligned} \tag{42}$$

Inverting eq. (41) allows us to convert perturbative expansions in the bare reaction rate, g, into expansions in the renormalised reaction rate, g_R. We find

$$g = g_R + g_*^{-1}(\epsilon) g_R^2 + \ldots. \tag{43}$$

The crucial point to all of this analysis is the following observation. Although for positive ϵ, g diverges as $L_\mu \to \infty$, rendering perturbative expansions in g useless for capturing the large mass behaviour of the theory, we will find that g_R remains finite as $\mu \to \infty$. Furthermore, g_R tends to a value which is of order ϵ. Therefore for small ϵ we can use eq. (43) to convert expansions in g into expansions in g_R which then have a better chance of remaining non-singular when we take $L_\mu \to \infty$. The process of replacing g with g_R is usually called *coupling constant renormalisation* in the literature.

The large mass behaviour of the renormalised reaction rate is determined by the β-function of the theory defined as

$$\beta(g_R) = \left(L_\mu \frac{\partial g_R}{\partial L_\mu} \right)\Big|_\lambda. \tag{44}$$

Using the fact that $L_\mu \frac{\partial}{\partial L_\mu} g^n = n\epsilon g^n$ together with eq. (41) and eq. (43) we quickly find

$$\beta(g_R) = \epsilon g_R(1 - g_*^{-1}(\epsilon)g_R + \ldots). \tag{45}$$

Eq. (44) now tells us how g_R changes as we vary L_μ. Solving this differential equation with the initial condition $g(L_0) = g_0$ determines how the reaction rate varies with scale. The behaviour is different in $d = 2$ and $d < 2$. In $d < 2$,

$$L_\mu \frac{\partial g}{\partial L_\mu} = \epsilon g(1 - \frac{g}{2\pi\epsilon}), \tag{46}$$

so that

$$g_R(L_\mu) = \frac{g_0 L_\mu^\epsilon}{(1 - \frac{g_0}{2\pi\epsilon})L_0^\epsilon + \frac{g_0}{2\pi\epsilon}L_\mu^\epsilon}. \tag{47}$$

We note that g_R goes to a fixed point value of $2\pi\epsilon$ (+corrections of $O(\epsilon^2)$) as $L_\mu \to \infty$ irrespective of the initial values of g_0 and L_0. For simplicity we can take $L_0 = (1 - \frac{g_0}{2\pi\epsilon})^{-1}$ giving

$$g_R(L_\mu) = \frac{g_0 L_\mu^\epsilon}{1 + \frac{g_0}{2\pi\epsilon}L_\mu^\epsilon}. \tag{48}$$

This universal behaviour as $L_\mu \to \infty$ is what is meant when we say that the renormalisation group flow has a perturbative fixed point in $d < 2$.

In $d = 2$, $\epsilon = 0$ so that

$$L_\mu \frac{\partial g}{\partial L_\mu} = -\frac{g^2}{2\pi}, \tag{49}$$

which gives

$$g_R(L_\mu) = \frac{g_0}{1 + \frac{g_0}{2\pi}\log\left(\frac{L_\mu}{L_0}\right)}. \tag{50}$$

Thus in $d = 2$ the reaction rate decays to 0 as $L_\mu \to \infty$ but logarithmically slowly and, unlike in the case $d < 2$, retains some memory of the small scale cut-off, L_0.

C. Average density in $d < 2$

Let us now show how all of this technology works by calculating the large mass behaviour of the average density. We define the renormalised density, $\langle R_\mu \rangle_{\mathrm{R}}$, by using eq. (43) to replace g with the renormalised reaction rate, g_R in eq. (37). Using eq. (42), a Taylor expansion shows that the replacement of g with g_R cancels the ϵ-singular term so we are left with

$$\langle R_\mu \rangle_{\mathrm{R}} = L_\mu^{\epsilon-2} \frac{1}{\sqrt{g_R}} \left[1 + o(g_R^2)\right]. \tag{51}$$

Now as $L_\mu \to \infty$, $g_R \to g_*$ so we can now take the limit to obtain

$$\langle R_\mu \rangle_{\mathrm{R}} \sim L_\mu^{\epsilon-2} \frac{1}{\sqrt{2\pi\epsilon}} \left[1 + o(\epsilon^2)\right]. \tag{52}$$

Finally note that as $\mu \to 0$,

$$L_\mu \sim \left(\frac{J\mu}{m_0}\right)^{-\frac{1}{d+2}},$$

allowing us to perform the inverse Laplace Transform required to return to mass space. Using the definition, eq. (13), of R_μ we finally find

$$N_m \underset{m\to\infty}{\sim} -\frac{1}{\Gamma\left(-d/d+2\right)} \frac{1}{\sqrt{2\pi\epsilon}} \left[1 + o(\epsilon^2)\right] m^{-\frac{2d+2}{d+2}}, \tag{53}$$

giving a scaling exponent which we know to be correct [9]. We recognise this as the K41 exponent. As discussed in section III we could have obtained this answer simply from dimensional arguments once we recognised that the reaction rate is renormalised away to infinity and hence cannot play any role in the answer. However this would ignore the possibility of anomalous dimensions. By calculating the one loop corrections to $\langle R_\mu \rangle$ we confirmed the absence of any relevant (for $\langle R_\mu \rangle$) couplings other than the reaction rate itself. We shall find that this is not the case for the higher order correlation functions.

D. Higher order moments of the density

A natural object to study to gather more information about the mass distribution function would be moments of the density of the form $M_n(m) = \langle N_m^n \rangle$. As explained in the appendix B (see eq. B18 and the explanation thereafter) these moments exhibit "extreme" anomalous scaling characterised by Burgers-like scalings :

$$\langle N_m^n \rangle \sim \langle N_m \rangle \; m \tilde{\to} \infty \; m^{-\frac{2d+2}{d+2}}. \tag{54}$$

For the MM however, this anomaly is somewhat trivial from a physical perspective. It arises because large masses become large by absorbing almost all nearby particles. Thus asymptotically, the number of heavy particles on a given lattice site ends up being either zero or one. Taking moments of such a distribution will always give the behaviour described by eq. (54). However, the analysis of appendix B which allows one to extract this essentially non-mean field behaviour from an initially weakly coupled theory is not trivial and can be expected to yield interesting results in other contexts. To observe true the multiscale structure of the mass model one should really study multipoint correlation functions. We do this next.

E. Higher order multi-point correlation functions in $d < 2$

The analysis for the higher order correlation functions is not quite so simple as for the density. By replacing g with g_R in eq. (36) we get

$$\langle R_{\mu_1} R_{\mu_2} \rangle_{g \to g_R} = \langle R_{\mu_1} \rangle_{\rm R} \langle R_{\mu_2} \rangle_{\rm R} \left[1 - \frac{g_R}{2\pi\epsilon} + o(g_R^2) \right]. \tag{55}$$

We see that we have removed the ϵ-singularities from the $\langle R_\mu \rangle$ factors but the singularity inside the square brackets remains. The correct definition of the renormalised 2-point function must include renormalisation of the amplitude of C_2, not just the reaction rate. This process is known as *composite operator renormalisation*. The correct definition of the renormalised 2-point function is therefore

$$\langle R_{\mu_1} R_{\mu_2} \rangle_{\rm R} = Z_2 \langle R_{\mu_1} R_{\mu_2} \rangle_{g \to g_R}, \tag{56}$$

where the amplitude Z_2 is chosen so that $\langle R_{\mu_1} R_{\mu_2} \rangle_{\rm R}$ is nonsingular in ϵ:

$$Z_2 = 1 + \frac{g_R}{2\pi\epsilon} + o(g_R^2). \tag{57}$$

The prefactor, Z_n, of the $n^{\rm th}$ order correlation function can be computed in a similar manner to the second order one. For example, in the loop expansion of the $3^{\rm rd}$ order correlation function, there are three diagrams containing singularities which are not removed by coupling constant renormalisation. These are shown in fig. 6. For the n-point function there are $\frac{1}{2}n(n-1)$ such diagrams. Each of these diagrams contributes $\frac{g_R}{2\pi\epsilon}$ to the one-loop expression for Z_n so that :

$$Z_n = 1 + \frac{1}{2} n(n-1) \frac{g_R}{2\pi\epsilon} + o(g_R^2). \tag{58}$$

This situation is a bit more complicated than before. To extract the scaling exponent we employ the technology of renormalisation group (RG) which was not truly necessary to compute the scaling of the density. Our discussion follows closely the presentation of [24]. The approach is based on the simple observation, already made at the end of section VI D, that the n^{th} order correlation function, $C^{(n)}(L_{\mu_1} \dots L_{\mu_n}) = \langle R_{\mu_1} \dots R_{\mu_n} \rangle$ does not depend on the arbitrary length scale L_μ, known in RG language as the *reference scale*. It immediately follows that

$$L_\mu \frac{\partial}{\partial L_\mu} \left(Z_n^{-1}(g_R)\, C_{\rm R}^{(n)}(L_{\mu_1} \dots L_{\mu_n}, g_R, L_\mu) \right) = 0.$$

The L_μ-dependence of the bracketed expression comes from three sources : an explicit dependence of $C_{\rm R}^{(n)}$ on L_μ, an implicit dependence through $g_R(L_\mu)$ and an implicit dependence through $Z_n(g_R(L_\mu))$. We can thus write

$$\left[L_\mu\frac{\partial}{\partial L_\mu}+L_\mu\frac{\partial g_R}{\partial L_\mu}\frac{\partial}{\partial g_R}+L_\mu\frac{\partial Z_n}{\partial L_\mu}\frac{\partial}{\partial Z_n}\right]Z_n^{-1}(g_R)\,C_{\rm R}^{(n)}(L_{\mu_1}\dots L_{\mu_n},g_R,L_\mu)=0, \tag{59}$$

where the partial derivative with respect to L_μ is now taken at fixed g_R and Z_n whose dependences on L_μ are catered for by the additional derivatives. This can then be arranged to give the equation :

$$\left[L_\mu\frac{\partial}{\partial L_\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_n(g_R)\right]C_{\rm R}^{(n)}(L_{\mu_1}\dots L_{\mu_n},g_R,L_\mu)=0, \tag{60}$$

where

$$\begin{aligned}\gamma(g_R) &= L_\mu\frac{\partial}{\partial L_\mu}\left(\log Z_2(g_R)\right),\\ &= \frac{g_R}{2\pi}+o(g_R^2),\end{aligned} \tag{61}$$

and $\beta(g_R)$ is given by eq. (45). By itself, this equation just tells us how $C_{\rm R}^{(n)}$ varies with physically meaningless reference scale, L_μ. However dimensional analysis provides extra information. Since the physical dimension of $C_{\rm R}^{(n)}$ is L^{-nd} it must satisfy an Euler equation [24]

$$\left[\sum_{i=1}^{n}L_{\mu_i}\frac{\partial}{\partial L_{\mu_i}}+L_\mu\frac{\partial}{\partial L_\mu}+nd\right]C_{\rm R}^{(n)}(L_{\mu_1}\dots L_{\mu_n},g_R,L_\mu)=0. \tag{62}$$

Suppose we now rescale all lengths by some amount, Λ, by introducing $\tilde{L}_{\mu_i}=\Lambda L_{\mu_i}$. Eq. (62) allows us to convert derivatives with respect to L_μ into derivatives with respect to Λ:

$$L_\mu\frac{\partial}{\partial L_\mu}C_{\rm R}^{(n)}(\tilde{L}_{\mu_1}\dots\tilde{L}_{\mu_n},g_R,L_\mu)=-\left(\Lambda\frac{\partial}{\partial\Lambda}+nd\right)C_{\rm R}^{(n)}(\tilde{L}_{\mu_1}\dots\tilde{L}_{\mu_n},g_R,L_\mu), \tag{63}$$

so that eq. (60) can be written as

$$\left[-\Lambda\frac{\partial}{\partial\Lambda}+\beta(g_R)\frac{\partial}{\partial g_R}-nd-\gamma_n(g_R)\right]C_{\rm R}^{(n)}(\tilde{L}_{\mu_1}\dots\tilde{L}_{\mu_n},g_R,L_\mu)=0. \tag{64}$$

This equation is called the Callan-Symanzic (C-S) equation. It tells us something physically useful, namely how the renormalised correlation function changes as we rescale its arguments by Λ. We wish to solve it in the limit of large Λ. This can be done using the method of characteristics. For $\Lambda=1$, $C_{\rm R}^{(n)}$ is given by the mean-field answer which is valid for small values of the $\tilde{L}_{\mu_i}$'s, thus providing an initial condition :

$$C_{\rm R}^{(n)}(\Lambda=1,g_R=g_0)=g_0^{-\frac{n}{2}}(L_{\mu_1}\dots L_{\mu_n})^{-d}. \tag{65}$$

For $d<2$, $\beta(g_R)=\epsilon g_R(1-\frac{g_R}{2\pi\epsilon})$ and the characteristic equations are

$$\frac{d\Lambda}{ds} = -\Lambda, \tag{66}$$

$$\frac{dg_R}{ds} = \epsilon g_R(1-\frac{g_R}{2\pi\epsilon}), \tag{67}$$

$$\frac{dC_{\rm R}^{(n)}}{ds} = \left(nd+\frac{1}{2}n(n-1)\frac{g}{2\pi}\right), \tag{68}$$

with the boundary conditions

$$\begin{aligned}\Lambda(g_0,s_0) &= 1,\\ g_R(g_0,s_0) &= g_0,\\ C_{\rm R}^{(n)}(g_0,s_0) &= g_0^{-\frac{n}{2}}(L_{\mu_1}\dots L_{\mu_n})^{-d}.\end{aligned} \tag{69}$$

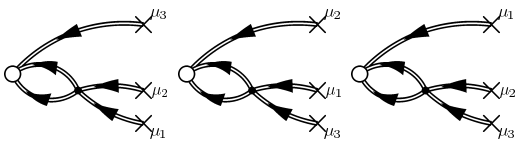

FIG. 6: Diagrams contributing to the anomalous dimension of the 3-point function $\langle R_{\mu_1} R_{\mu_2} R_{\mu_3} \rangle$ at the one loop level.

If we solve these equations, use the uniqueness of the characteristic curves to express s_0 and g_0 in terms of Λ and g_R and then evaluate the solution at $s = 0$, the answer can be found explicitly:

$$C_{\rm R}^{(n)}(\Lambda, g_R) = \left(\sqrt{\frac{1}{2\pi\epsilon}\left(1 - \left(1 - \frac{2\pi\epsilon}{g_R}\right)\Lambda^{-\epsilon}\right)} \right)^n (\Lambda L_{\mu_1} \dots \Lambda L_{\mu_n})^{-d} \left(\frac{\Lambda^{-\epsilon} - \left(1 - \frac{2\pi\epsilon}{g_R}\right)\Lambda^{-\epsilon}}{1 - \left(1 - \frac{2\pi\epsilon}{g_R}\right)\Lambda^{-\epsilon}} \right)^{\frac{1}{2}n(n-1)} . \tag{70}$$

Taking $\Lambda \to \infty$ we conclude

$$C_{\rm R}^{(n)}(\tilde{L}_{\mu_1} \dots \tilde{L}_{\mu_n}, g_R, L_\mu) \sim \prod_{i=1}^{n} L_{\mu_i}^{\frac{1}{2}\epsilon(n-1)} \sqrt{\frac{1}{2\pi\epsilon}}\, \tilde{L}_i^{-d-\frac{1}{2}\epsilon(n-1)}, \tag{71}$$

independent of the value of g_R. This independence is the consequence of the presence of a fixed point of the β-function. All values of g_R flow to the fixed-point value, $g^* = 2\pi\epsilon$, leaving a universal answer in the limit of large Λ. It remains to perform the inverse Laplace Transform to find the scaling properties of the original mass-space correlation functions.

To do this we note from eq. (33) that for large values of the $\tilde{L}_i$,

$$\tilde{L}_i = \left(\frac{J\tilde{\mu}_i}{D m_0} \right)^{-\frac{1}{d+2}} . \tag{72}$$

It is then easy to perform the n inverse Laplace Transforms with respect to the $\tilde{\mu}_i$ to get

$$C_{\rm R}^{(n)}(\tilde{m}_1 \dots \tilde{m}_n, g_R, L_\mu) \sim \prod_{i=1}^{n} \tilde{m}_i^{-\frac{2d+2}{d+2} - \frac{\epsilon(n-1)}{2(d+2)}}. \tag{73}$$

The mass scaling of $C_{\rm R}^{(n)}$ is therefore $m^{-\gamma_n}$ with

$$\gamma_n = n\frac{2d+2}{d+2} + \frac{n(n-1)\epsilon}{2(d+2)} + o(\epsilon^2). \tag{74}$$

Note that γ_n acquires a correction to the value predicted from K41 theory signalling the breakdown of self-similarity in low dimensions. This is the multiscaling curve against which we compared our numerical results in fig. 2.

F. Logarithmic Corrections in $d = 2$

In $d = 2$ scale invariance is broken by the presence of logarithmic corrections to the mean field scaling. For completeness, let us calculate the powers of the logarithms acquired by the $C_R^{(n)}$'s. In $d = 2$, $\beta(g_R) = -\frac{g_R^2}{2\pi}$ and the C-S equation, eq. (64), reads :

$$\left[-\Lambda\frac{\partial}{\partial\Lambda} - \frac{g_R^2}{2\pi}\frac{\partial}{\partial g_R} - 2d - \frac{1}{2}n(n-1)\frac{g_R}{2\pi} \right] C_{\rm R}^{(n)}(\tilde{L}_{\mu_1} \dots \tilde{L}_{\mu_n}, g_R, L_\mu) = 0. \tag{75}$$

The initial condition is again given by the mean-field answer which in $d = 2$ is

$$C_{\rm R}^{(n)}(\Lambda = 1, g_R = g_0) = g_0^{-\frac{n}{2}} (L_{\mu_1} \dots L_{\mu_n})^{-2}. \tag{76}$$

The characteristic equations are

$$\frac{d\Lambda}{ds} = -\Lambda, \tag{77}$$

$$\frac{dg_R}{ds} = -\frac{g_R^2}{2\pi}, \tag{78}$$

$$\frac{dC_{\rm R}^{(n)}}{ds} = \left(2n + \frac{1}{2}n(n-1)\frac{g}{2\pi}\right), \tag{79}$$

with the boundary conditions as in eq. (69) with d replaced by 2. These can again be solved explicitly at $s = 0$ to give:

$$C_n^{\rm R}(\Lambda, g_R) = \prod_{i=1}^{n} \sqrt{\frac{1}{g_R} + \frac{1}{2\pi}\log\Lambda}\,(\Lambda L_{\mu_i})^{-2} \left(\frac{2\pi}{g}\right)^{\frac{1}{2}(n-1)} \left(\frac{2\pi}{g} + \log\Lambda\right)^{-\frac{1}{2}(n-1)}, \tag{80}$$

or, in terms of the rescaled lengths, $\tilde{L}_i$:

$$C_n^{\rm R}(\tilde{L}_1 \ldots \tilde{L}_n, g_R, L_\mu) = \prod_{i=1}^{n} \sqrt{\frac{1}{g_R} + \frac{1}{2\pi}\log\frac{\tilde{L}_i}{L_{\mu_i}}}\,\tilde{L}_i^{-2} \left(\frac{2\pi}{g}\right)^{\frac{1}{2}(n-1)} \left(\frac{2\pi}{g} + \log\frac{\tilde{L}_i}{L_{\mu_i}}\right)^{-\frac{1}{2}(n-1)}. \tag{81}$$

The large mass limit corresponds to $\tilde{L}_i/L_{\mu_i} \to \infty$ in which case

$$C_n^{\rm R}(\tilde{L}_1 \ldots \tilde{L}_n, g_R, L_\mu) \sim \prod_{i=1}^{n} \sqrt{\frac{1}{2\pi}\log\frac{\tilde{L}_i}{L_{\mu_i}}}\,\tilde{L}_i^{-2} \left(\frac{2\pi}{g}\right)^{\frac{1}{2}(n-1)} \left(\log\frac{\tilde{L}_i}{L_{\mu_i}}\right)^{-\frac{1}{2}(n-1)}. \tag{82}$$

To recover the asymptotic behaviour in mass space, it is again necessary to take inverse Laplace transforms with respect to the $\tilde{\mu}_i$ as $\tilde{\mu}_i \to 0$. Recalling the definition, eq. (33) of L_μ we can write

$$C_n^{\rm R}(\tilde{m}_1 \ldots \tilde{m}_n, g_R) \sim \int \prod_{i=1}^{n} e^{-\tilde{m}_i\tilde{\mu}_i} d\tilde{\mu}_i \left(\sqrt{-\frac{1}{8\pi}\log m_i\tilde{\mu}_i}\,\sqrt{\frac{J\tilde{\mu}_i}{Dm_0}}\right)^n \left(\frac{2\pi}{g}\right)^{\frac{1}{2}n(n-1)} \left(-\frac{1}{4}\log m_i\tilde{\mu}_i\right)^{-\frac{1}{2}n(n-1)}. \tag{83}$$

By introducing scaling variables $x_i = \tilde{m}_i\tilde{\mu}_i$ and keeping leading order terms in $\log\tilde{m}_i/m_i$ the asymptotic behaviour of this integral as the $\tilde{m}_i \to \infty$ is shown to be

$$C_n^{\rm R}(\tilde{m}_1 \ldots \tilde{m}_n, g_R) \sim C(g_R) \prod_{i=1}^{n} \left(\sqrt{\frac{J}{D}\,\log\frac{\tilde{m}_i}{m_i}}\,\tilde{m}_i^{-\frac{3}{2}}\right) \left(\log\frac{\tilde{m}_i}{m_i}\right)^{-\frac{1}{2}(n-1)}. \tag{84}$$

The density, $n = 1$, picks up a square root of a logarithm coming from the renormalisation of the reaction rate. However the higher order correlation functions pick up *additional* logarithmic corrections which come from the anomalous dimension of the two point function. Note that in $d = 2$ the asymptotic behaviour retains some memory of the low mass cut-offs, m_i. Furthermore the prefactor, denoted above by $C(g_R)$ remains dependent on the value of g_R, unlike in $d < 2$.

VIII. RENORMALISED SMOLUCHOWSKI EQUATION

Although the correspondence between constant kernel aggregation and the $A + A \to A$ system is convenient for calculations and greatly simplifies the field theoretic description of the problem, the physics is sometimes obscured by this mapping. In this section we show that the results for the average mass density, derived in section VII C from a field theoretic perspective, can also be interpreted in a more physically transparent way using some heuristic arguments closely related to the so-called Smoluchowski approximation used in [25] to take into account fluctuations in low dimensional heterogeneous annihilation.

The idea is as follows. If we are interested in the average mass density, then the only diagrams remaining in the field theory associated with the stochastic rate equation, eq. (14), after all trees have been summed are those which renormalise the reaction rate. Therefore it should be possible to understand the problem directly in terms of a

standard Smoluchowski kinetic theory with a suitably modified kernel, without reference to the Laplace-transformed field R_μ. Furthermore, the stationary state of this renormalised Smoluchowski theory can be found directly without need for any R.G. analysis using a technique, known as the Zakharov Transformation, borrowed from the the theory of wave turbulence.

The variation of the dimensionless reaction rate, g, as the length scale L_μ is changed is determined by the β-functions computed in section VII B. From eq. (48) and eq. (50) we can obtain the renormalisation law for the physical reaction rate, $\lambda_R(L_\mu) = g_R(L_\mu)\, L_\mu^{-\epsilon}$. Since the Laplace variable, μ, can be thought of as an inverse mass, we can make the heuristic substitution,

$$L_\mu \sim \left(\frac{J}{Dm_0}\right)^{-\frac{1}{4}} m^{\frac{1}{d+2}}, \tag{85}$$

to motivate the following mass-dependent reaction rates :

$$\lambda_R(m) = \frac{\lambda}{1 + \frac{\lambda}{2\pi\epsilon}\left(\frac{J}{m_0 D}\right)^{-\frac{\epsilon}{d+2}} m^{\frac{\epsilon}{d+2}}}, \quad d < 2, \tag{86}$$

$$\lambda_R(m) = \frac{\lambda}{1 + \frac{\lambda}{8\pi}\ln\left(\frac{m}{m_0}\right)}, \qquad d = 2. \tag{87}$$

The renormalised Smoluchowski equation (RSE) is then obtained from the mean field Smoluchowski equation by substituting the above mass-dependent reaction rate for λ. The density should therefore satisfy the following equation at large times

$$\begin{aligned}
\frac{\partial N(m,t)}{\partial t} &= \int_0^\infty dm_1 dm_2 \lambda_R(m) N(m_1,t) N(m_2,t) \delta(m - m_1 - m_2) \\
&- \int_0^\infty dm_1 dm_2 \lambda_R(m_2) N(m,t) N(m_1,t) \delta(m_2 - m - m_1) \\
&- \int_0^\infty dm_1 dm_2 \lambda_R(m_1) N(m,t) N(m_2,t) \delta(m_1 - m_2 - m) \\
&+ \frac{J}{m_0}\,\delta(m - m_0).
\end{aligned} \tag{88}$$

The stationary state of this equation is best studied using the method of Zakharov transformations as detailed in [10, 26]. For $d < 2$ and $m \gg m_0$, $\lambda_R(m) = 2\pi\epsilon (J/D)^{\frac{\epsilon}{d+2}} m^{-\frac{\epsilon}{d+2}}$. The constant flux solution obtained by applying the Zakharov transformation to eq. (88) with this kernel is

$$N(m) = c_K \left(\frac{J}{D}\right)^{\frac{d}{d+2}} m^{-\frac{2d+2}{d+2}}. \tag{89}$$

The constant, c_K, can be calculated exactly. However since we have only calculated the renormalised kernel heuristically, it does not make sense to do so at this point. Note that we recover the predictions of the K41 theory for the large mass behaviour of the solution to the RSE. Conversely, for small masses we see that $\lambda_R(m) = \lambda$. In this limit, the Zakharov transformation gives the original mean field solution for constant kernel aggregation,

$$N(m) = \sqrt{\frac{J}{4\pi\lambda}} m^{-\frac{3}{2}}, \tag{90}$$

as found already from our RG analysis. It is an easy task to show that both the mean field and renormalised mean field density distributions are local in the sense that the inertial range mass transfer in the stationary state does not depend strongly on the source (or the sink which removes large masses if one is present). Therefore both are physically realisable solutions and both carry the same flux, J, of mass from small masses to large. The complete stationary distribution for constant kernel aggregation should therefore exhibit a crossover from the mean field solution to the renormalised mean field solution at a mass, m_c which is given by $(J/D)^{\frac{\epsilon}{d+2}} m_c^{-\frac{\epsilon}{d+2}} \approx 1$. We note that this crossover from a mean field spectrum to a fluctuation dominated one has been conjectured to occur in other turbulent systems, in particular in wave turbulence [27], although these systems do not readily lend themselves to such systematic analysis.

In $d = 2$, analysis of the RSE equation allows us to obtain the logarithmic correction to the mean field spectrum, again without resort to R.G. arguments. Strictly speaking, the Zakharov transformation technology of [10, 26] only works for homogeneous kernels which leaves us with the question of what to do with the logarithm in eq. (87). It turns out that the approach is easily adapted to extract the logarithmic correction to the spectrum at large masses. We give a brief outline here. In $d = 2$, the renormalised interaction, eq. (87), behaves for large masses as

$$\lambda_R(m) \sim \frac{1}{\ln\left(\frac{m}{m_0}\right)}. \tag{91}$$

After application of the Zakharov transformations described in [10], the stationary RSE, eq. (88), it becomes

$$\begin{aligned} 0 \;=\; & \int_0^\infty dm_1 dm_2 \left[\ln\left(\frac{m}{m_0}\right)^{-1} N(m_1)\, N(m_2) \right. \\ & - \ln\left(\frac{m^2}{m_2 m_0}\right)^{-1} N(m)\, N(\frac{m m_1}{m_2}) \left(\frac{m}{m_2}\right)^2 \\ & \left. - \ln\left(\frac{m^2}{m_1 m_0}\right)^{-1} N(m)\, N(\frac{m m_2}{m_1}) \left(\frac{m}{m_1}\right)^2 \right] \\ & \delta(m - m_1 - m_2). \end{aligned} \tag{92}$$

Let us now look for a solution of the form $N(m) = c_K \ln(m/m_0)^y m^{-x}$. This substitution yields a rather messy expression which we analyse by introducing new integration variables, μ_1, μ_2 defined by $m_1 = m\mu_1$, $m_2 = m\mu_2$ and then expanding the resulting expression as a power series in $\ln(m/m_0)^{-1}$. After some algebra one obtains :

$$\begin{aligned} 0 \;=\; & c_K^2 m^{2-2x} \int_0^\infty d\mu_1 d\mu_2 (\mu_1\mu_2)^{-x} \left[\ln\left(\frac{m}{m_0}\right)^{2y-1} \right. \\ & - \ln\left(\frac{m}{m_0}\right)^{2y-1} \mu_1^{2x-2} - \ln\left(\frac{m}{m_0}\right)^{2y-1} \mu_2^{2x-2} \\ & \left. + \;\mathrm{O}\left(\ln\left(\frac{m}{m_0}\right)^{-1}\right) \right] \; \delta(1 - \mu_1 - \mu_2). \end{aligned} \tag{93}$$

It is clear that the leading logarithms cancel out asymptotically as $m \to \infty$ if we choose $y = 1/2$. The integrand then vanishes asymptotically for $x = 3/2$ as in the usual mean field case. The renormalised Kolmogorov spectrum in $d = 2$ for large masses is therefore

$$P(m) = c_K \sqrt{\ln\left(\frac{m}{m_0}\right)}\, m^{-\frac{3}{2}}, \tag{94}$$

as found from the RG analysis of section VII F.

Let us close this section by discussing the connection between the renormalised Smoluchowski equation and the Smoluchowski approximation used in [25] to study the kinetics of heterogeneous annihilation. The essence of the argument used in [25] is as follows. Consider a heterogeneous system of annihilating particles where the reactants have a continuous distribution of diffusivities with the slower particles being less probable than the faster ones. Now consider the reaction between particles with diffusivity D and "slower" particles with diffusivity $D' < D$. Since the slower particles are rare we can neglect reactions between slow particles. We therefore estimate the effective reaction rate for the slow particles by considering each slow particle to be stationary in a uniform background cloud of faster particles and calculating the diffusive flux of fast particles reaching the slow particle from the background cloud. One estimates the effective reaction rate in d dimensions at time t to be $(D + D')^{d/2} t^{-1+d/2}$. Substitution of this effective reaction rate in the mean field rate equation constitutes the Smoluchowski approximation which greatly improves the estimate of the asymptotic decay rate of the particle density in $d < 2$. In the case at hand, the dynamics naturally generates a distribution of particle masses in which heavy particles are much rarer than the lighter ones. So, although in our model all particles have equal diffusivity, D, in the reference frame of a heavy particle we can consider the main interaction to be with a uniform background cloud of light particles. Thus one can envisage an effective reaction rate for the large mass particles of $(2D)^{d/2} t^{-1+d/2}$. However in our case the mass flux into the system is a constant, J, so that the system reaches a stationary state in which the natural unit of time

for particles of mass m is $t \sim (J^{-1}D^{-d/2}m)^{2/(d+2)}$. Thus the time dependent effective reaction rate of [25] should be replaced in the stationary state of the constant kernel aggregation problem with a mass dependent reaction rate $\lambda_R(m) \sim D^{d/2}(JD^{d/2})^{-(d-2)/(d+2)}m^{(d-2)/(d+2)}$ which we recognise as the large mass behaviour of the renormalised kernel, eq. (86). Thus the large mass asymptotics of the RSE corresponds to the correction of the mean field rate equation by the Smoluchowski approximation. The field theory approach with which we derived the original results allows unambiguous identification of the set of diagrams which have been summed to give this approximation.

IX. CONCLUSIONS

To summarise, the most important result of this work is the following fact : the steady state mass density PDF of cluster-cluster aggregation in the presence of a steady source of monomers exhibits non-trivial multiscaling in dimensions less than or equal to 2. Technically, this means that the exponents, ζ_n, describing the large mass asymptotics of the density correlation functions, $C_n(m_1 \dots m_n) = \langle N_{m_1} \dots N_{m_n} \rangle$, depend nonlinearly on the order, n, of the correlation function. Physically it means that the system is in a regime where strong correlations (or in this case, anticorrelations) between particles dominate the statistics putting the problem outside of the domain of applicability of mean-field theory. In the early sections of the article we established the presence of multiscaling more-or-less rigorously by calculating the asymptotic scaling behaviour of the C_n for $n = 0, 1, 2$ and remarking that they do not lie on one line. The latter part of the article is devoted to detailing a nonrigorous derivation of ζ_n for general n as an expansion in $\epsilon = 2 - d$ using renormalisation group techniques. To leading order in ϵ, the scaling curve is quadratic in n. In $d = 2$ the power law corrections to the mean-field exponents become logarithmic. Our results are interesting both from the perspective of particle systems and from the perspective of turbulence. In the context of aggregation, the understanding of the detailed role of fluctuations beyond their effect on the density is still embryonic. The effect of fluctuations on the density can be taken into account using the so-called Smoluchowski approximation. In the closing section of the article we showed how our approach is connected to this approximation. Whereas the Smoluchowski approximation is only useful for the density, our methods work for arbitrary correlation functions. The connection to the theory of turbulence stems from the analogy between the mass cascade in the stationary state of cluster-cluster aggregation and the energy cascade in the stationary state of homogeneous isotropic turbulence. Throughout the article we have tried to emphasise both the usefulness and limitations of this analogy. Both systems exhibit nontrivial multiscaling. In the case of turbulence, this multiscaling has proven to be very difficult to understand quantitatively. One might hope that the analogy developed here might, if viewed in the correct way, provide some insight. At present, however, this is merely an aspiration.

Acknowledgments

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APPENDIX A: EFFECTIVE HAMILTONIAN OF MASS MODEL.

A microstate of the mass model is given by the vector of occupation numbers, $\vec{N} = \{N_{\vec{x},m}\}_{\vec{x}\in\mathbf{Z}^d, m\in\mathbf{Z}^+}$. The probability measure $\mathbf{P}_t(\vec{N})$ on the space of microstates satisfies the Master Equation, which follows from the dynamics of the mass model:

$$\frac{\partial}{\partial t}P_t(\vec{N}) = \frac{D}{2d}\sum_m \sum_{\langle \vec{x},\vec{x}'\rangle} \Big((N_{\vec{x},m}+1)P_t(\{\dots n_{\vec{x},m}+1\dots n_{\vec{x}',m}-1\dots\}) - N_{\vec{x},m}P_t(\vec{N})\Big)$$

$$+\frac{\lambda}{2}\sum_{\vec{x}} \sum_{m=m_1+m_2} \Big((N_{\vec{x},m_1}+1)(N_{\vec{x},m_2}+1)P_t(\{\dots N_{\vec{x},m_1}+1\dots N_{\vec{x},m_2}+1\dots N_{\vec{x},m}-1\dots\}) - N_{\vec{x},m_1}N_{\vec{x},m_2}P_t(\vec{N})\Big)$$

$$\sum_{\vec{x}}\sum_m \left(\frac{J}{m_0}\delta(m-m_0)P_t(\{\dots N_{\vec{x},m}-1\dots\}) - \frac{J}{m_0}\delta(m-m_0)P_t(\vec{N})\right).$$

The first line of the Master Equation describes diffusion, second - aggregation, the third - deposition. By "..." we denote components of the microstate, which are the same on the right hand side as on the left hand side. Let $|P_t\rangle = \sum_{\vec{N}} P_t(\vec{N}) \prod_{\vec{x},m} (a^\dagger_{\vec{x},m})^{N_{\vec{x},m}}|0\rangle$ be the state vector corresponding to probability measure P_t.

The evolution equation for $|P_t\rangle$ follows by differentiating $|P_t\rangle$ with respect to time and using the master equation to express the result back in terms of $|P_t\rangle$. A simple calculation shows that

$$\partial_t|P_t\rangle = -H_{tm}|P_t\rangle, \tag{A1}$$

where H_{tm} is an effective Hamiltonian of the mass model given by :

$$\begin{aligned} H_{tm} = D\sum_m \sum_{\langle\vec{x},\vec{x}'\rangle} (a^\dagger_{\vec{x},m} - a^\dagger_{\vec{x}',m})(a_{\vec{x},m} - a_{\vec{x}',m}) \\ -\lambda\sum_{\vec{x}} \sum_{m,m_1,m_2} \delta(m - m_1 - m_2) a^\dagger_{\vec{x},m} a_{\vec{x},m_1} a_{\vec{x},m_2} \\ +\lambda\sum_{\vec{x}} \sum_{m_1,m_2} a^\dagger_{\vec{x},m_1} a^\dagger_{\vec{x},m_2} a_{\vec{x},m_1} a_{\vec{x},m_2} \\ -\sum_{\vec{x},m} \frac{J}{m_0}\delta(m - m_0)(a^\dagger_{\vec{x},m} - 1) \end{aligned} \tag{A2}$$

where $\langle\vec{x},\vec{x}'\rangle$ denotes summation over pairs of nearest neighbours. Note that we replaced λ with 2λ in order to simplify the notation. The Hamiltonian operator, eq. (A2), already includes the "shift" introduced to simplify the computation of correlation functions as described in appendix B.

APPENDIX B: RELATION BETWEEN CORRELATION FUNCTIONS OF INTERACTING PARTICLE SYSTEMS AND STOCHASTIC RATE EQUATIONS.

In this appendix, we establish the relation between interacting particle systems and stochastic rate equations. To avoid unnecessary complications, we start with an example of a zero-dimensional system. We then show how the zero dimensional results get modified in $d > 0$ using the example of the mass model.

The microstate of a zero dimensional particle system is determined by the number of particles present at time t. Let the probability that there are N particles left at time t be $P_t(N)$. If the particle dynamics is Markovian, this probability satisfies the linear master equation which is first order in time. The principle step of Doi's map is to rewrite the master equation as an evolution equation for a certain state vector in the Fock space spanned by microstates of the particle system. The Fock space $\mathcal{F}$ can be introduced as follows. Let $|N\rangle$ be an element (a "state") of $\mathcal{F}$, corresponding to a microstate with N particles, $N = 0, 1, 2, \ldots$. The state $|0\rangle$ is often referred to as 'vacuum'. Then $\mathcal{F}$ is defined as a space of all linear combinations of states $|N\rangle$ with complex coefficients. Let $a^\dagger$ be the creation operator acting in $\mathcal{F}$. By definition,

$$a^\dagger|N\rangle = |N+1\rangle \tag{B1}$$

The annihilation operator, a, acting in $\mathcal{F}$ is defined as follows:

$$[a, a^\dagger] = 1, \tag{B2}$$
$$a|0\rangle = 0, \tag{B3}$$

where $[A, B] \equiv AB - BA$ denotes the commutator of the operators $A,\ B$. It then follows that

$$a|N\rangle = N|N-1\rangle. \tag{B4}$$

It is also convenient to define the left vacuum $\langle 0|$ (an element of the space dual to $\mathcal{F}$) by means of the following relations:

$$\begin{aligned} \langle 0|0\rangle = 1, \\ \langle 0|a^\dagger = 0. \end{aligned} \tag{B5}$$

Consider the following element of $\mathcal{F}$:

$$|P_t\rangle = \sum_{N=0}^{\infty} P_t(N)|N\rangle. \tag{B6}$$

The state $|P_t\rangle$ can be used to generate all probabilities $P_t(N)$, as $P_t(N) = \frac{1}{N!}\langle 0|a^N|P_t\rangle$. The vector $|P_t\rangle$ satisfies an equation, which follows from the master equation for probabilities:

$$\frac{d}{dt}|P_t\rangle = -H|P_t\rangle, \tag{B7}$$

where H is an evolution operator (Hamiltonian). The exact form of the Hamiltonian follows from the microscopic rules of evolution. For example, if particles undergo pairwise annihilations at rate λ, then (see, for example, [18])

$$H = -\lambda(a^2 - a^{\dagger\ 2}a^2). \tag{B8}$$

The first term on the right hand side of (B8) describes annihilation, while the second term comes from the "minus" term in the master equation, which accounts for the probability of non-reaction. The formal solution to (B7) is

$$|P_t\rangle = e^{-tH}|P_0\rangle, \tag{B9}$$

where the state $|P_0\rangle$ is determined by the initial distribution of particles. The problem of solving Master Equation is now reduced to solving an effective Schroedinger Equation in imaginary time, which can be often done using powerful methods of quantum/statistical field theory.

Another important observation of Doi's theory is the averaging formula. Let $Z_t(J)$ be the generating function of moments of the probability distribution $P_t(N)$. By definition, $Z_t(J) \equiv E(e^{JN_t}) = \sum_0^\infty e^{JN}P_t(N)$. Using the definitions above, one can prove the following:

$$Z_t(J) = \langle 0|e^a e^{Ja^\dagger a}|P_t\rangle. \tag{B10}$$

To show this, expand the right hand side to give

$$\begin{aligned} Z_t(J) &= \sum_{N=0}^\infty \sum_{K=0}^\infty P_t(N)\frac{J^K}{K!}\langle 0|e^a(a^\dagger a)^K|N\rangle \\ &= \sum_{N=0}^\infty \sum_{K=0}^\infty \frac{J^K}{K!}N^K P_t(N)\langle 0|e^a|N\rangle \\ &= \sum_{N=0}^\infty \sum_{K=0}^\infty \frac{J^K}{K!}N^K P_t(N) \\ &= \sum_{N=0}^\infty e^{JN}P_t(N) \equiv E(e^{JN}). \end{aligned}$$

In the proof we used the fact that $\hat{N} \equiv a^\dagger a$ is the operator measuring the number of particles, $\hat{N}|N\rangle = N|N\rangle$, and the identity $e^a a^{\dagger\ N} = (a^\dagger + 1)^N e^a$. The form of the correlation function on the right hand side of eq. (B10) is not very suitable for practical computations due to the presence of the shift operator e^a inside the brackets. This problem can be overcome by commuting e^a to the right and using $e^a|0\rangle = |0\rangle$. It follows from the commutation relations [eq. (B2)] that $e^a O(a^\dagger, a) = O(a^\dagger + 1, a)$ for any operator $O(a^\dagger, a)$. Using this fact one finds that

$$Z_t(J) = \langle 0|e^{J(a^\dagger+1)a}e^{-t\tilde{H}}|\tilde{P}_0\rangle, \tag{B11}$$

where $\tilde{H}(a^\dagger, a) = H(a^\dagger + 1, a)$, $|\tilde{P}_0\rangle = e^a|P_0\rangle$. The expression $\langle 0|e^{J(a^\dagger+a)a}$ can be simplified further. Note that $[a^\dagger a, a] = -a$. In other words, the operators a and $a^\dagger a$ form a basis of a Lie algebra isomorphic to a subalgebra of $sl(2)$ consisting of upper triangular matrices. Consequently, the Campbell-Hausdorff formula implies that

$$e^{J(a^\dagger+a)a} = e^{Ja^\dagger a}e^{f(J)a}, \tag{B12}$$

where $f(J)$ is a function to be determined. Differentiating both sides of eq. (B12) with respect to J, commuting all operators multiplying exponents in the derivatives to the right, and comparing both sides of the resulting equality, we find a differential equation for f:

$$f'(J) = f(J) + 1, \tag{B13}$$

which should be solved with the boundary condition $f(0) = 0$. The answer is $f(J) = e^J - 1$. Taking into account that $\langle 0|e^{Ja^\dagger a} = \langle 0|$, we find

$$Z_t(J) = \langle 0|e^{(e^J-1)a} e^{-t\tilde{H}}|\tilde{P}_0\rangle, \tag{B14}$$

The right hand side of this relation can be rewritten as a path integral using the Trotter formula. Assume for simplicity that initial probability distribution of the number of particles $P_0(t)$ is Poisson with intensity N_0. Then,

$$Z_t(J) = \int\int \prod_\tau da(\tau)da(\tau)e^{(e^J-1)a(t)}e^{-S_{eff}(t)}, \tag{B15}$$

where the integration is performed over the space of complex paths $(\bar{a}(\tau), a(\tau))_{\tau\in[0,t]}$ and

$$S_{eff}(t) = \int_0^t d\tau\Big(\bar{a}(\tau)\partial_\tau a(\tau) + \tilde{H}(\bar{a}(\tau), a(\tau)) - N_0\bar{a}(\tau)\Big) \tag{B16}$$

is an effective 'action functional'. Here $\tilde{H}(\bar{a}(\tau), a(\tau))$ is the symbol of operator $\tilde{H}(a^\dagger, a)$, which is assumed to be normally ordered. We conclude that

$$E(e^{JN_t})_{P_t} = \langle e^{(e^J-1)a(t)}\rangle_{S_{eff}(t)}, \tag{B17}$$

where we stressed by our notations, that the averaging in the left hand side of eq. (B17) is performed over the space of microstates using probability distribution P_t, while averaging in the right hand side is performed over the space of all paths $(\bar{a}(\tau), a(\tau))_{\tau\in[0,t]}$ using the functional measure $e^{-S_{eff}(t)}$. Differentiating both sides of eq. (B17) with respect to $\theta = e^J - 1$ and setting $\theta = 0$ we find that moments of $a(t)$ correspond to *factorial* moments of P_t:

$$E\big(N_t(N_t-1)(N_t-2)...(N_t-k+1)\big)_{P_t} = \langle a(t)^k\rangle_{S_{eff}(t)}, k = 1, 2, 3, \dots \tag{B18}$$

In particular $E(N_t) = \langle a(t)\rangle, E(N_t^2) = \langle a(t)^2 + a(t)\rangle$, etc.

Eq. (B18) allows one to capture strong non-mean field behaviour of an interacting particle system even if the field theory characterised by eq. (B16) is well approximated by mean field theory. The simplest example is as follows. Assume that $\langle a(t)^k\rangle \approx \langle a(t)\rangle^k$, but $E(N_t) = \langle a(t)\rangle \ll 1$. Using these assumptions and eq. (B18) to evaluate moments of N_t, we find that $E(N^k) = E(N)$, which is essentially non-mean field behaviour.

For particle systems with pairwise local interactions, eqs. (B17) and (B18) can be formulated in terms of stochastic differential equations thus avoiding references to non-rigorous path integral constructions. To illustrate this, consider the reaction $A + A \to \emptyset$ in zero dimensions. The effective Hamiltonian is given by eq. (B8), $\tilde{H} \equiv H(a^\dagger + 1, a) = \lambda(2a^\dagger a^2 + a^{\dagger\ 2}a^2)$. The corresponding functional integral measure can be rewritten using the Hubbard-Stratonovich identity as

$$e^{-\int_0^t d\tau\left(\bar{a}(\tau)\partial_\tau a(\tau)+\lambda(2a^\dagger a^2+a^{\dagger\ 2}a^2)-N_0\bar{a}(\tau)\right)} = \prod_{\tau'} d\xi(\tau')e^{-\frac{1}{2}\int_0^t d\tau\xi^2(\tau)}$$
$$\times\ e^{-\int_0^t d\tau\left(\bar{a}(\tau)\partial_\tau a(\tau)+2\lambda a^\dagger a^2-N_0\bar{a}(\tau)-i\sqrt{2\lambda}\xi(\tau)a^\dagger(\tau)a(\tau)\right)}. \tag{B19}$$

The field $\xi(t)$ is standard Gaussian white noise. Note that the expression in the exponent in the second line of eq. (B19) is linear in the field $a^\dagger$, which allows it to be integrated out. This results in a δ-functional with an argument

$$\partial_\tau a(\tau) + 2\lambda a^2 - N_0\delta(\tau) - i\sqrt{\lambda}\xi(\tau)a(\tau) = 0, \tag{B20}$$

in which we recognise a rate equation of $A + A \to \emptyset$ augmented by imaginary multiplicative noise term (Lee-Cardy equation).

We can now interpret eqs. (B17) and (B18) as follows. Let N_t be the number of particles left after time t in the system of annihilating particles. Let $a(t)$ be the solution to stochastic differential eq. (B20). Then eqs. (B17) and (B18) are true, given that $\langle ...\rangle_{S_{eff}}$ denotes averaging over the white noise $\xi(t)$ [29]. In the stated form, the correspondence between the Markov chain describing $A + A \to \emptyset$ system and the Lee-Cardy stochastic partial differential equation can be proven rigorously and extended to a large class of interacting particle systems in $d \geq 0$ [28].

Here we will only give a path integral derivation of the relation between correlation functions in the mass model in $d > 0$ and correlation functions of Stochastic Smoluchowski equation, eq. (11).

First, consider the case of discrete space and mass. A microstate of the mass model is specified by stating the number of particles of a given mass at a given site. In other words, the microstate is a vector $\{N_{\vec{x},m}\}_{\vec{x}\in Z^d, m\in Z_+}$ with non-negative integer components. The operator of mass distribution is

$$\widehat{N}_{\vec{x},m} = a^\dagger_{\vec{x},m} a_{\vec{x},m}, \tag{B21}$$

where $a^\dagger_{\vec{x},m}$ and $a_{\vec{x},m}$ are creation and annihilation operators satisfying the following commutation relations:

$$[a^\dagger_{\vec{x},m}, a_{\vec{x}',m'}] = \delta_{\vec{x},\vec{x}'}\delta_{m,m'}. \tag{B22}$$

Similarly to the zero dimensional case considered above, the generating functional for mass distribution correlation functions can be written as follows:

$$\begin{aligned} Z_t[\vec{J}] &\equiv E\left(e^{\sum_{\vec{x},m} J_{\vec{x},m} N_{t,\vec{x},m}}\right), \\ &= \langle 0| e^{\sum_{\vec{x},m} a_{\vec{x},m}} e^{\sum_{\vec{x},m} J_{\vec{x},m}\widehat{N}_{\vec{x},m}} e^{-H_{\rm MM}} |P_0\rangle, \end{aligned} \tag{B23}$$

where

$$\begin{aligned} H_{\rm MM} &= D\sum_m \sum_{\langle \vec{x},\vec{x}'\rangle} (a^\dagger_{\vec{x},m} - a^\dagger_{\vec{x}',m})(a_{\vec{x},m} - a_{\vec{x}',m}) \\ &-\lambda \sum_{xv} \sum_{m,m_1,m_2} \delta(m - m_1 - m_2) a^\dagger_{xv,m} a_{\vec{x},m_1} a_{\vec{x},m_2} \\ &+\lambda \sum_{\vec{x}} \sum_{m_1,m_2} a^\dagger_{\vec{x},m_1} a^\dagger_{\vec{x},m_2} a_{\vec{x},m_1} a_{\vec{x},m_2} \\ &-\sum_{\vec{x},m} \frac{J}{m_0}\delta(m - m_0)(a^\dagger_{xv,m} - 1) \end{aligned} \tag{B24}$$

is an effective Hamiltonian of the mass model and P_0 is an initial probability measure on the space of microstates. An outline of the derivation of $H_{\rm MM}$ is presented in appendix A.

As before, we need to commute the projection operator $exp\sum_{\vec{x},m} a_{\vec{x},m}$ to the right. Computations, which are completely analogous to those performed in 0-dimensional case give:

$$Z_t[\vec{J}] = \langle 0| e^{\sum_{\vec{x},m} (exp(J_{\vec{x},m}) - 1) a_{\vec{x},m}} e^{-\tilde{H}_{\rm MM}} |\tilde{P}_0\rangle, \tag{B25}$$

where $\tilde{H}_{\rm MM}$ is obtained from $H_{\rm MM}$ by the shift $a^\dagger_{\vec{x},m} \to a^\dagger_{\vec{x},m} + 1$ and $|\tilde{P}_0\rangle = e^{\sum_{\vec{x},m} a_{\vec{x},m}} |P_0\rangle$. The right hand side of eq. (B25) can be re-written in path integral form in the conventional way. A lengthy but straightforward calculation shows that

$$\begin{aligned} Z_t[\vec{J}] &= \int \prod_{\vec{x}',m',\tau'} d\bar{\phi}(\vec{x}', m', \tau') d\phi(\vec{x}', m', \tau') \\ &\quad e^{\sum_{\vec{x},m} (exp(J_{\vec{x},m}) - 1)\phi(\vec{x},m,t)} e^{-S_{\rm MM}(t)}, \end{aligned} \tag{B26}$$

where

$$\begin{aligned} S_{\rm MM}(t) &\equiv \int_0^t dt \left(\sum_{\vec{x},m} \bar{\phi}(\vec{x}, m, t)\partial_t \phi(\vec{x}, m, t) + \tilde{H}_{\rm MM}[\bar{\phi}, \phi] \right) \\ &= \int_0^t dt \sum_{\vec{x},m} \left(\bar{\phi}(\vec{x}, m, t) \left[\left(\frac{\partial}{\partial t} - D\Delta \right) \phi(\vec{x}, m, t) \right.\right. \\ &\quad -\lambda \sum_{m'} \phi(\vec{x}, m', t)\phi(\vec{x}, m - m', t) \\ &\quad \left.\left. +2\lambda \phi(\vec{x}, m, t) \sum_{m'} \phi(\vec{x}, m', t) - \frac{J}{m_0}\delta_{m,m_0} \right] \right. \\ &\quad \left. +\lambda \sum_{m'} \bar{\phi}(\vec{x}, m, t)\phi(\vec{x}, m, t)\bar{\phi}(\vec{x}, m', t)\phi(\vec{x}, m', t) \right), \end{aligned} \tag{B27}$$

where Δ is a discrete Laplacian and J is the rate of input of mass into the system. Note that the expression in square brackets in the right hand side of eq. (B27) is just the constant kernel Smoluchowski equation. The last term in $S_{\rm MM}$ accounts for all correlation effects. The exponential of this term can be rewritten using the Hubbard-Stratonovich transformation as follows:

$$\begin{aligned} e^{-\lambda\int_0^t dt\sum_{\vec{x}}\left(\sum_m \bar{\phi}(\vec{x},m,t)\phi(\vec{x},m,t)\right)^2} &= \int\prod_{\vec{x}',\tau'} d\xi(\vec{x}',\tau')e^{-\frac{1}{2}\int_0^t dt\sum_{\vec{x}}\xi(\vec{x},t)^2} \\ & e^{i\sqrt{2\lambda}\int_0^t dt\sum_{\vec{x},m}\xi(\vec{x},t)\bar{\phi}(\vec{x},m,t)\phi(\vec{x},m,t)}. \end{aligned} \tag{B28}$$

Note that the field ξ is Gaussian, uncorrelated both in space and time. Using eq. (B28), the functional measure of integration in eq. (B26) can be rewritten in the form

$$e^{-S_{\rm MM}} = \int\prod_{\vec{x}',\tau'} d\xi(\vec{x}',\tau')e^{-\frac{1}{2}\int_0^t dt\xi^2(\vec{x},t)}e^{-\int_0^t\sum_{\vec{x},m}\bar{\phi}(\vec{x},m,t)L[\phi,\xi]}, \tag{B29}$$

where

$$\begin{aligned} L[\phi,\xi] &= \left(\frac{\partial}{\partial t}-D\Delta\right)\phi(m)-\lambda\sum_{m'=0}^{m}\phi(\vec{x},m',t)\phi(\vec{x},m-m',t) \\ &+2\lambda\phi(\vec{x},m,t)\sum_{m'=0}^{\infty}\phi(\vec{x},m',t)-\frac{J}{m_0}\delta_{m,m_0}-i\sqrt{2\lambda}\phi(m)\eta(\vec{x},t). \end{aligned} \tag{B30}$$

The exponent in the right hand side of eq. (B29) is linear in $\bar{\phi}$. Hence the path integral over fields $\bar{\phi}$, ϕ and ξ localises to paths satisfying Euler-Lagrange equation

$$\frac{\delta}{\delta\bar{\phi}(\vec{x},m,t)}\int_0^t dt\sum_{\vec{x},m}\bar{\phi}(\vec{x},m,t)L[\phi,\xi], \tag{B31}$$

or

$$L[\phi,\xi]=0, \tag{B32}$$

which is a discrete version of the Smoluchowski equation, eq. (11). For the sake of clarity, we restate the result concerning the relation between the mass model and stochastic Smoluchowski equation here. In order to calculate the generating functional of density correlation functions in the mass model [eq. (B23)], one has to solve the Stochastic Smoluchowski equation, eq.(B32), for $\phi[\xi](\vec{x},m,t)$, then average $e^{\sum_{\vec{x},m}(exp(J_{\vec{x},m})-1)\phi[\xi](\vec{x},m)}$ with respect to Gaussian white noise ξ. In other words,

$$Z_t(\vec{J}) = E\left(e^{\sum_{\vec{x},m}(exp(J_{\vec{x},m})-1)\phi[\xi](\vec{x},m)}\right)_\xi \tag{B33}$$

Our final task is to discuss the modification and consequences of eq. (B33) in the continuous limit. The latter is taken according to the following set of rules:

$$\begin{aligned} \vec{x} &\to \frac{\vec{x}}{a_x}, \\ m &\to \frac{m}{a_m}, \\ \phi(\frac{\vec{x}}{a_x},\frac{m}{a_m},t) &\to \frac{1}{a_x^d a_m}\phi(\vec{x},m,t), \\ N(\frac{\vec{x}}{a_x},\frac{m}{a_m},t) &\to \frac{1}{a_x^d a_m}N(\vec{x},m,t), \\ D &\to \frac{D}{a_x^2}, \\ J &\to \frac{Ja_x^d}{a_m}, \\ m_0 &\to \frac{m_0}{a_m}, \end{aligned} \tag{B34}$$

where a_x and a_m are lattice cut-offs in $\vec{x}$- and m-spaces respectively. The continuous limit is obtained by performing replacements eq. (B34) in eqs. (B30) and (B33) and taking lattice cut-offs to zero while keeping other parameters fixed. As a result one recovers the stochastic Smoluchowski equation (SSE) [eq. (11)].

Note that the continuous field theory equivalent to SSE is renormalizable in dimensions two and less, therefore eq. (B34) is justified in these dimensions only. The continuous counterpart of eq. (B33) is

$$Z_t(\vec{J}) = E\biggl(e^{\int\int d\vec{x}dm(exp(J(\vec{x},m))-1)\phi[\xi](\vec{x},m)}\biggr)_\xi \tag{B35}$$

APPENDIX C: EXPRESSION OF PROBABILITY OF MULTI-PARTICLE CONFIGURATIONS IN TERMS OF SOLUTIONS TO SSE.

In this appendix, we show how probability of multiparticle configurations can be calculated from SSE. Equation (B35) leads to the following relation between correlation functions:

$$\begin{aligned} E\biggl(\phi(\vec{x},m,t)\biggr)_\xi &= E\biggl(N_t(\vec{x},m)\biggr), \\ E\biggl(\phi(\vec{x}_1,m_1,t)\phi(\vec{x}_2,m_2,t)\biggr)_\xi &= E\biggl(N_t(\vec{x}_1,m_1)N_t(\vec{x}_2,m_2) \\ &\quad -\delta^d(\vec{x}_1-\vec{x}_2)\delta(m_1-m_2)N_t(\vec{x}_1,m_1)\biggr)_\xi, \end{aligned} \tag{C1}$$

and so on. Suppose that we are interested in the statistics of the total number of particles $\Delta N_t(\vec{x},m)$ in a volume element ΔV centred around $\vec{x}$ with masses in the interval $[m, m+\Delta m]$. In terms of the local mass distribution,

$$\Delta N_t(\vec{x},m) = \int_{\Delta V} d^dx' \int_m^{m+\delta m} dm' N_t(\vec{x}',m'). \tag{C2}$$

Let

$$\Delta\phi_t(\vec{x},m) = \int_{\Delta V} d^dx' \int_m^{m+\delta m} dm' \phi_t(\vec{x}',m'). \tag{C3}$$

Integrating eq. (C1) with respect to mass and space, we find

$$\begin{aligned} E\biggl(\Delta\phi(\vec{x},m,t)\biggr) &= E\biggl(\Delta N_t(\vec{x},m,t)\biggr) \\ E\biggl(\Delta\phi^2(\vec{x},m,t)\biggr) &= E\biggl(\Delta N_t(\vec{x},m,t)(\Delta N(\vec{x},t,m)-1)\biggr) \\ &\cdots \\ E\biggl(\Delta\phi^n(\vec{x},m,t)\biggr) &= E\biggl(\prod_{k=0}^{n-1}(\Delta N(\vec{x},t,m)-k)\biggr), \end{aligned} \tag{C4}$$

which is a multi-dimensional counterpart of 0-dimensional result of eq. (B18).

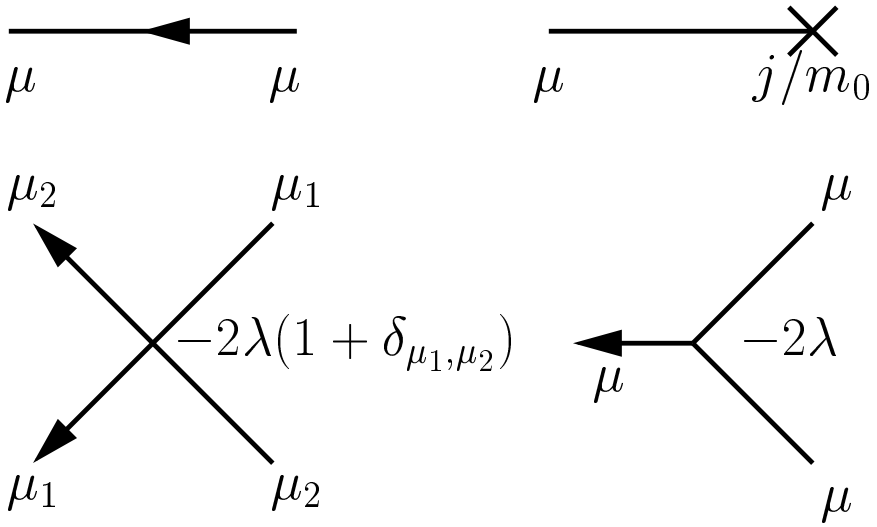
In this paper we study scaling properties of probability of finding multiple particles of large mass in $\Delta V\Delta m$. Density of such particles is low. Thus, factorial moments entering the right hand side of eq. (C4) can be estimated in the limit of large mass m as follows:

$$\begin{aligned} E\biggl(\prod_{k=0}^{n-1}(\Delta N(\vec{x},t,m)-k)\biggr) &\equiv \sum_{p=n}^{\infty}\biggl(\prod_{k=0}^{n-1}(p-k)\biggr)Prob\biggl(\vec{x}_1,\ldots,\vec{x}_p\in\Delta V; m_1,\ldots,m_p\in[m,m+\Delta m]\biggr) \\ &\approx n!Prob\biggl(\vec{x}_1,\ldots,\vec{x}_n\in\Delta V; m_1,\ldots,m_p\in[m,m+\Delta m]\biggr) \end{aligned} \tag{C5}$$

Combining this result with eq. (C4), we obtain desired relation between probabilities of multi-particle configurations and moments of solutions to SSE in the limit of large masses:

$$Prob\biggl(\vec{x}_1,\ldots,\vec{x}_n\in\Delta V; m_1,\ldots,m_p\in[m,m+\Delta m]\biggr) = \frac{1}{n!}E\biggl(\Delta\phi^n(\vec{x},m,t)\biggr)_\xi \tag{C6}$$

[1] U. Frisch, *Turbulence : the Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[2] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Modern Phys. **73**, 913 (2001).
[3] V. Zakharov, V. Lvov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992).
[4] C. Connaughton, R. Rajesh, and O. Zaboronski, Phys. Rev. Lett. **94**, 194503 (2005).
[5] A. E. Scheidegger, Bull. IASH. **12**, 15 (1967).
[6] P. S. Dodds and D. H. Rothman, Phys. Rev. E **59**, 4865 (1999).
[7] H. Takayasu, Phys. Rev. Lett. **63**, 2563 (1989).
[8] G. Huber, Physica A **170**, 463 (1991).
[9] R. Rajesh and S. Majumdar, Phys. Rev. E **62**, 3186 (2000).
[10] C. Connaughton, R. Rajesh, and O. Zaboronski, Phys. Rev. E **69**, 061114 (2004).
[11] P. L. Krapivsky, J. F. F. Mendes, and S. Redner, Phys. Rev. B **59**, 15950 (1999).
[12] S. N. Coppersmith, C. h Liu, S. Majumdar, O. Narayan, and T. A. Witten, Phys. Rev. E **53**, 4673 (1996).
[13] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. E **61**, 6337 (2000).
[14] R. Rajesh, Phys. Rev. E **69**, 036128 (2004).
[15] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[16] M. Doi, J. Phys. A **9**, 1465 (1976).
[17] M. Doi, J. Phys. A **9**, 1479 (1976).
[18] J. Cardy, *Field theory and nonequilibrium statistical mechanics*, notes are available at the website http://www-thphys.physics.ox.ac.uk/users/JohnCardy/.
[19] B. Lee, J. Phys. A **27**, 2633 (1994).
[20] O. Zaboronski, Phys. Lett. A **281**, 119 (2001).
[21] P. Martin, E. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).
[22] L. Peliti, J. Phys. A **19**, L365 (1986).
[23] M. Droz and L. Sasvári, Phys. Rev. E **48**, R2343 (1993).
[24] J. Binney, N. Dowrick, A. Fisher, and M. Newman, *The Theory of Critical Phenomena* (Clarendon Press, Oxford, 1992), chap. 11 contains an excellent pedagogical introduction to the renormalisation group in the context of the Landau-Ginzberg model, much of which is equally applicable to the problem under study here.
[25] P. Krapivsky, E. Ben-Naim, and S. Redner, Phys. Rev. E **50**, 2474 (1994).
[26] V. Kontorovich, Physica D **152-153**, 676 (2001).
[27] L. Biven, S. Nazarenko, and A. Newell, Phys. Lett. A **280**, 28 (2001).
[28] R. Tribe and O. Zaboronski, *Interacting particle systems and lee-cardy equations*, in preparation.
[29] In deriving this statement one also has to check that the value of the determinant of the functional derivative of the left hand side of eq. (B20) with respect to ϕ is one. This is true in forward time regularization, which we always implicitly assume.


μ
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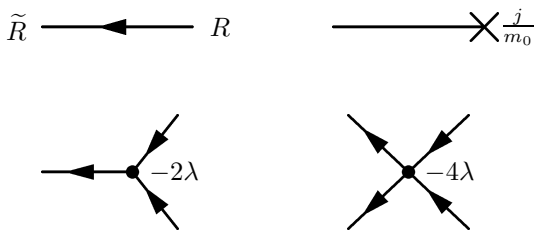
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R
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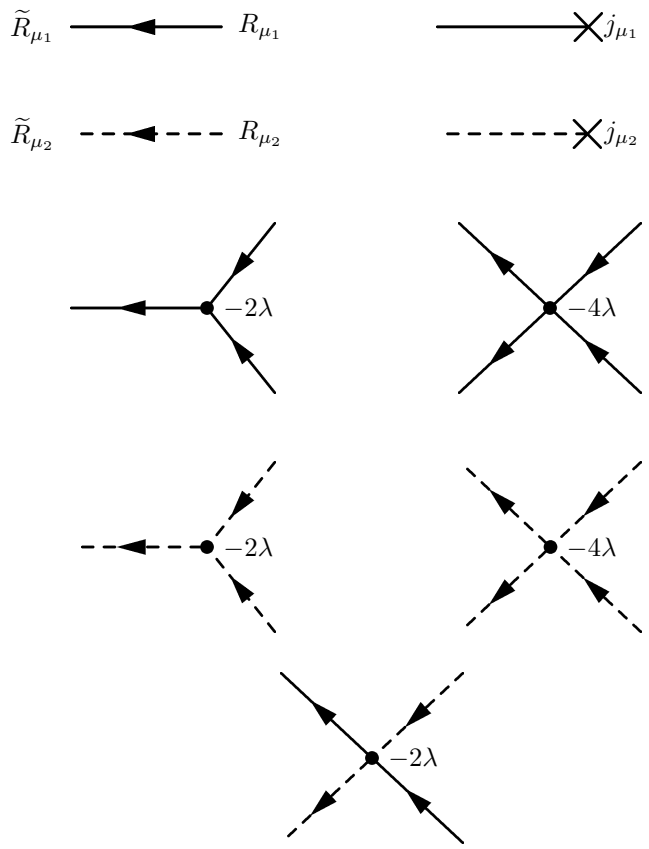
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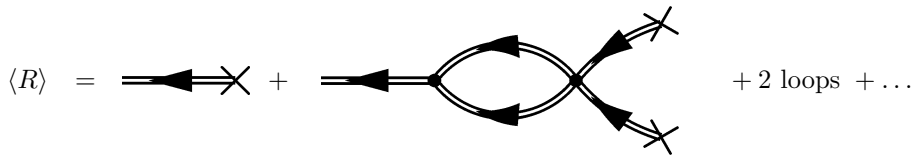
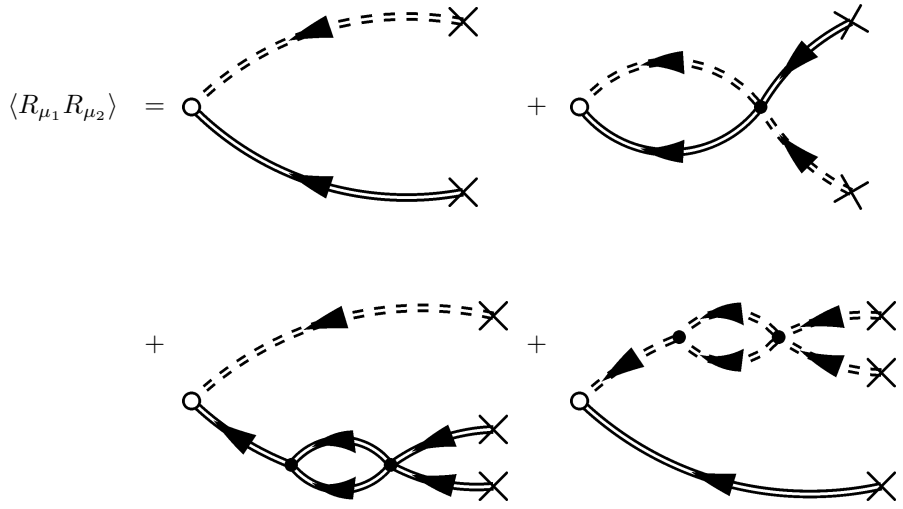
-2λ

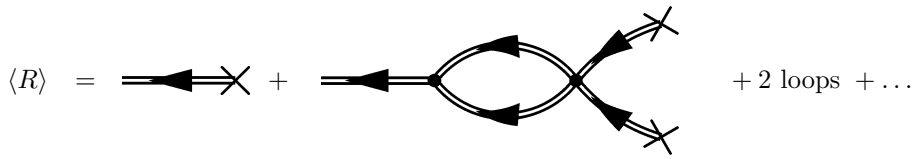
(A)



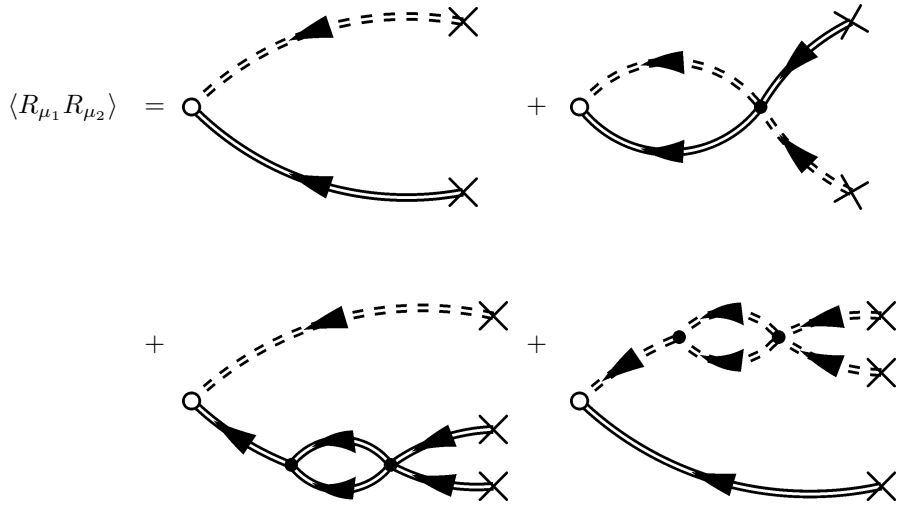
(B)

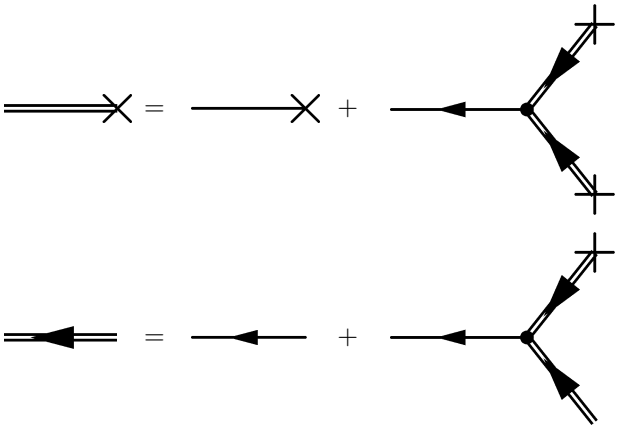


(A)

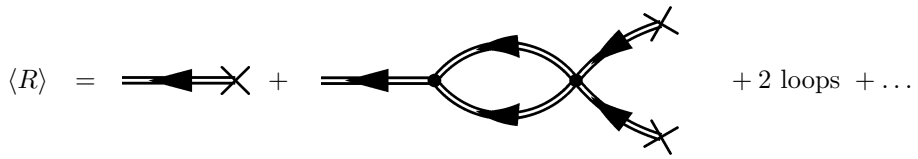


(B)





(A)



(B)